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PARAMETRIZING UNSTABLE AND VERY UNSTABLE MANIFOLDS

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Dedicated to Yulij Ilyashenko on the occasion of his 60th birthday

Abstract. Existence and uniqueness theorems for unstable manifolds are well-known. Here we prove certain refinements. Let $f\colon (\mathbb{C}^n, 0) \to \mathbb{C}^n$ be a germ of an analytic diffeomorphism, whose derivative $Df(0)$ has eigenvalues $\lambda_1, \ldots, \lambda_n$ such that

$$|\lambda_1| \geq \cdots \geq |\lambda_k| > |\lambda_{k+1}| \geq \cdots \geq |\lambda_n|,$$

with $|\lambda_k| > 1$.

Then there is a unique k-dimensional invariant submanifold whose tangent space is spanned by the generalized eigenvectors associated to the eigenvalues $\lambda_1, \ldots, \lambda_k$, and it depends analytically on f.

Further, there is a natural parametrization of this "very unstable manifold," which can be extended to an analytic map $\mathbb{C}^k \to \mathbb{C}^n$ when f is defined on all of $\mathbb{C}^n$, and is an injective immersion if f is a global diffeomorphism.

We also give the corresponding statements for stable manifolds, which are analogous locally but quite different globally.

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The origin of this paper lies in an attempt to numerically compute unstable manifolds for Hénon mappings $H\colon \mathbb{C}^2 \to \mathbb{C}^2$. Suppose $H(\boldsymbol{p}) = \boldsymbol{p}$ and that $DH(\boldsymbol{p})$ has eigenvalues λ, μ with $|\lambda| > 1 > |\mu|$. Let $\boldsymbol{v}$ be an eigenvector for the eigenvalue λ. Our parametrization of the unstable manifold at $\boldsymbol{p}$ is given by the following theorem.

Theorem 1. *The limit*

$$\Phi(z) = \lim_{m\to\infty} H^m\left(\boldsymbol{x} + \frac{z}{\lambda^m}\boldsymbol{v}\right) \tag{1}$$

exists and gives an injective immersion of $\mathbb{C}$ onto the unstable manifold $W^u(\boldsymbol{p})$ that satisfies $\Phi(\lambda z) = H(\Phi(z))$.

This theorem is proved as a special case of Corollary 25. The parametrization is quite effectively computable, and is used in the program *saddle drop* [3], explained

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in [2]. As far as I know this construction of unstable manifolds, obvious though it is, is not in the literature; it may represent a more flexible approach than the graph-transform method, described in many papers and books on dynamical systems. On the other hand, there is nothing new about taking a "model map" g and a "perturbed map" f, and constructing a conjugacy by considering $\lim_{m\to\infty} f^m \circ g^{-m}$. I learned of the construction in Nelson's book [5]; it is sometimes called the *scattering construction*, since a similar construction arises in physics in scattering theory.

A case in point comes up in the paper [1] by Buzzard, Hruska and Ilyashenko, in which they prove that Hénon mappings satisfying Axiom A form a residual set of parameters in the parameter space of Hénon maps. Generalizing this to higher dimensions requires the "very unstable manifold" given below, specifically in the case where this is not the unstable manifold. I don't see how to get this from graph transforms.

We will work throughout in the complex analytic category, because that is the context in which the problems arose, and because many of the proofs are simpler in that case. But it should be clear that most of the results (particularly Theorem 6, on which everything else is based) go through verbatim in the C^∞ category. In the case of finite differentiability, it might be delicate to figure out just how differentiable the parametrizations of the very unstable manifolds are.

This paper contains a large number of statements; by far the main result is Theorem 6. Everything else follows from this result and a number of formal power series computations, which are more or less standard.

I wish to thank Professor Ilyashenko for some very useful conversations, and especially the referee for pointing out a serious error in an earlier version of the paper.

2. Polynomial Changes of Variables

At several points, we will change variables to bring our mappings to a favorable form. In these sorts of computations, half the battle is choosing notation that is sufficiently light to be readable and sufficiently precise to be unambiguous.

We will use multiindices $I \in \mathbb{N}^n$ to label monomial mappings $\mathbb{C}^n \to \mathbb{C}$: if $I = (i_1, \ldots, i_n) \in \mathbb{N}^n$, we define

$$\boldsymbol{x}^I = x_1^{i_1} \cdots x_n^{i_n}, \quad \text{of total degree} \quad |I| = i_1 + \cdots + i_n.$$

We will use the same notation when we have eigenvalues $\lambda_1, \ldots, \lambda_n$ by setting $\boldsymbol{\lambda}^I = \lambda_1^{i_1} \cdots \lambda_n^{i_n}$.

Our real interest is not scalar functions but mappings $\mathbb{C}^n \to \mathbb{C}^n$, and we define the monomial mapping

$$p_I^j(\boldsymbol{x}) = \begin{bmatrix} 0 \\ \vdots \\ \boldsymbol{x}^I \\ \vdots \\ 0 \end{bmatrix} \leftarrow j\text{th position.}$$

The mappings p_I^j are totally ordered by setting

$$p_I^j \prec p_{I'}^{j'} \quad \text{if} \quad \begin{cases} |I| < |I'|, \text{ or} \\ |I| = |I'| \text{ but } I \neq I' \text{ and } i_l > i'_l \text{ for the first } l \text{ such that } i_l \neq i'_l, \text{ or} \\ I = I' \text{ and } j > j'. \end{cases} \tag{2}$$

Our main tool in making polynomial changes of variables will be the mappings $\Phi_I^j(a)\colon \mathbb{C}^n \to \mathbb{C}^n$ defined by

$$\Phi_I^j(a)(\boldsymbol{x}) = \boldsymbol{x} + ap_I^j(\boldsymbol{x}).$$

If $|I| > 1$, the map $\Phi_I^j(a)(\boldsymbol{x})$ is tangent to the identity at $\mathbf{0}$, hence a local diffeomorphism. Whenever I does not include the index j, it is even a global diffeomorphism. This case arises in Proposition 4, but the fact that the changes of coordinates are global diffeomorphisms is not used in the proof.

We chose our order on the mappings p_I^j so as to keep track of changes when passing from a mapping $f\colon \mathbb{C}^n \to \mathbb{C}^n$ to its conjugate by $\Phi_I^j(a)$.

Proposition 2. *Let U be a neighborhood of $\mathbf{0}$ in $\mathbb{C}^n$ and let $f\colon U \to \mathbb{C}^n$ be an analytic mapping such that $f(\mathbf{0}) = \mathbf{0}$ and such that the matrix of $Df(\mathbf{0})$ is upper triangular, with the eigenvalues $\lambda_1, \ldots, \lambda_n$ appearing in that order on the diagonal. If the coefficient of p_I^j in the Taylor series of f at the origin is b, then the coefficient of the same monomial p_I^j in the Taylor series of*

$$g = (\Phi_I^j(a))^{-1} \circ f \circ (\Phi_I^j(a))$$

is $b + a(\lambda_j - \boldsymbol{\lambda}^I)$.

Moreover, the other monomials whose coefficients are different for f and g all appear later in the order of monomial mappings.

Proof. First observe that

$$(\Phi_I^j(a))^{-1} = \Phi_I^j(-a) + o(|\boldsymbol{x}|^{|I|}).$$

Next, observe that only the linear terms of f (i. e., the triangular matrix $Df(\mathbf{0})$) and the term bp_I^j can contribute terms of degree $\leq |I|$ that are different for f and g (see for instance Proposition 3.4.4 of [4]), and recall that terms of degree $> |I|$ appear later in the order. Thus we only need to consider the mapping

$$\Phi_I^j(-a) \circ (Df(\mathbf{0}) + bp_I^j) \circ \Phi_I^j(a).$$

Since $Df(\mathbf{0})$ is upper triangular, $(Df(\mathbf{0}) + bp_I^j)$ and $(Df(\mathbf{0}) + bp_I^j) \circ \Phi_I^j$ differ only by terms $ap_I^{j'}$ for $j' \geq j$, corresponding to the non-zero terms in the jth column of $Df(\mathbf{0})$. In particular, the coefficient of p_I^j in $(Df(\mathbf{0}) + bp_I^j)$ is $b + a\lambda_j$.

Any such terms with $j' > j$ will still appear in $\Phi_I^j(-a) \circ Df(\mathbf{0}) \circ \Phi_I^j$. There may also be new terms in the jth row, but they all appear later in the order, except for p_I^j itself, which appears with coefficient $b + a(\lambda_j - \boldsymbol{\lambda}^I)$. $\square$

Example 5 below illustrates this procedure.

3. Prepared Mappings

In proving the local existence of unstable manifolds, we will need the conditions of Equation (4) below to be satisfied; this is only true if we make an appropriate change of variables and bring the mapping to *prepared form*.

Suppose $U \subset \mathbb{C}^n$ is a neighborhood of the origin, and $f\colon U \to \mathbb{C}^n$ is an analytic mapping with $f(\mathbf{0}) = \mathbf{0}$. Let the eigenvalues of $Df(\mathbf{0})$ be $\lambda_1, \ldots, \lambda_n$ with

$$|\lambda_1| \geq \cdots \geq |\lambda_k| > |\lambda_{k+1}| \geq \cdots \geq |\lambda_n| \quad \text{and} \quad |\lambda_k| > 1. \tag{3}$$

We will refer to $\lambda_1, \ldots, \lambda_k$ as the *high* eigenvalues, and $\lambda_{k+1}, \ldots, \lambda_n$ as the *low* eigenvalues. Let $E^H \subset \mathbb{C}^n$ be spanned by the generalized eigenspaces of the high eigenvalues, and let $E^L \subset \mathbb{C}^n$ be the subspace spanned by the generalized eigenspaces for the low eigenvalues.

Let $\mathrm{pr}^H\colon \mathbb{C}^n \to E^H$ be the projection onto E^H, parallel to E^L and $\mathrm{pr}^L\colon \mathbb{C}^n \to E^L$ be the projection onto E^L, parallel to E^H. A basis $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ of C^n will be called an (H, L)-basis if $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k$ is a basis of E^H and $\boldsymbol{v}_{k+1}, \ldots, \boldsymbol{v}_n$ is a basis of E^L.

Definition 3. We will say that f is (N, k)-prepared if, written in coordinates with respect to an (H, L)-basis, the power series expansion of f at the origin contains no monomial terms p_I^j with $|I| \leq N, \ j > k$ and only the indices $1, \ldots, k$ appearing in I.

Equivalently, f is (N, k)-prepared if there exists $\epsilon > 0$ and a constant C such that for any $\boldsymbol{x} \in E^H$ with $|\boldsymbol{x}| < \epsilon$, we have $|\,\mathrm{pr}^L(f(\boldsymbol{x}))| \leq C|\boldsymbol{x}|^N$.

The second characterization shows that being (N, k)-prepared is independent of the (H, L)-basis chosen.

Proposition 4 shows that we lose very little by assuming that our mappings are appropriately prepared.

Proposition 4. *Let U be a neighborhood of 0 in $\mathbb{C}^n$, and let $f\colon U \to \mathbb{C}^n$ be an analytic mapping with $f(\mathbf{0}) = \mathbf{0}$. Suppose the eigenvalues $\lambda_1, \ldots, \lambda_n$ of $Df(\mathbf{0})$ satisfy the inequalities of Equation (3). Choose N so that $|\lambda_k|^N > |\lambda_1|$. Then there exists an invertible polynomial mapping $\Phi\colon \mathbb{C}^n \to \mathbb{C}^n$ such that $\Phi^{-1} \circ f \circ \Phi$ is (N, k)-prepared.*

Proof. Begin by putting $Df(\mathbf{0})$ in triangular form by a linear change of variables, so that $\lambda_1, \ldots, \lambda_n$ appear on the diagonal in that order.

We will say that the term bp_I^j is (N, k)-*offending* if $|I| \leq N$, $j > k$ and only $i_1, \ldots, i_k$ appear in I. We will take the (N, k)-offending terms in the order on monomial mappings of Equation (2). Let p_I^j be the first. According to Proposition 2, the coefficient of p_I^j in $(\Phi_I^j(a))^{-1} \circ f \circ \Phi_I^j(a)$ is $b + a(\lambda_j - \boldsymbol{\lambda}^I)$, and since $|\lambda_j| < |\boldsymbol{\lambda}^I|$, we can choose a so that this coefficient vanishes. The only terms of $(\Phi_I^j(a))^{-1} \circ f \circ \Phi_I^j(a)$ that are different from those of f appear later in the order, so we can now eliminate the next (N, k)-offending term, until all the (N, k)-offending terms are eliminated.

Observe that all the $\Phi_I^j(a)$ used in the process are globally invertible. Indeed, if $\Phi_I^j(\boldsymbol{x}) = \boldsymbol{u}$, then obviously $x_i = u_i$ for $i \neq j$. The equation for x_j reads $x_j + a\boldsymbol{x}^I = u_j$, and since j is not among the indices appearing in I, this gives $x_j = u_j - a\boldsymbol{u}^I$. $\square$

Example 5. Let

$$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \lambda^3 x \\ \lambda^2 y + z \\ \lambda z + bxy \end{pmatrix}.$$

We will try to bring f to be (2, 2)-prepared, and at the moment the term bxy in the third line is (2, 2)-offending. Thus we will conjugate the mapping by $\Phi^3_{[1,1,0]}(a)$ for an appropriate a. Note that

$$(\Phi^3_{[1,1,0]}(a))^{-1} = \Phi^3_{[1,1,0]}(-a);$$

the analogous statement will always be true when eliminating (N, k)-offending terms.

Let us compute the conjugation:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \overset{\Phi}{\mapsto} \begin{pmatrix} x \\ y \\ z + axy \end{pmatrix} \overset{f}{\mapsto} \begin{pmatrix} \lambda^3 x \\ \lambda^2 y + z + axy \\ \lambda(z + axy) + bxy \end{pmatrix} \overset{\Phi^{-1}}{\longmapsto}$$
$$\begin{pmatrix} \lambda^3 x \\ \lambda^2 y + z + axy \\ \lambda(z + axy) + bxy - a(\lambda^3 x)(\lambda^2 y + z + axy) \end{pmatrix}.$$

The term that really matters is $xy(b + a(\lambda - \lambda^5))$ in the third coordinate; this term can be made to vanish by setting $a = b/(\lambda^5 - \lambda)$. The other new terms of degree 2 are

(1) the term $a\lambda^3 xz$ in the third line; this is later than the term xy in the third term because of the second condition in the definition of $\prec$, and
(2) the term axy in the second line; this is also later than the xy-term in the third line by the third condition in the definition of $\prec$.

There is also a cubic term in the third line, which is also later. In fact, after conjugation there are no more (2, 2)-offending terms: the term in x^2y in the third term involves only x and y but has too high degree. (If we required a (3, 2)-prepared mapping, we would need to eliminate it.)

4. Local Existence of Very Unstable Manifolds

Suppose that U is a neighborhood of $\mathbf{0}$ in $\mathbb{C}^n$, and that $f : U \to \mathbb{C}^n$ is an analytic mapping with $f(\mathbf{0}) = \mathbf{0}$. Let the eigenvalues $\lambda_1, \ldots, \lambda_n$ of $Df(\mathbf{0})$ be ordered to satisfy both conditions of Equation (3).

By a linear change of variables, we may choose an (H, L)-basis so that the matrix of $Df(0)$ is upper triangular.

Choose N so that $|\lambda_k|^N > |\lambda_1|$. By a further polynomial change of variables tangent to the identity, we may assume that f is (N, k)-prepared.

Define

$$g = \mathrm{pr}^H \circ (f|_{E^H}) \colon E^H \to E^H.$$

We obtain g from the power series of f by retaining only the monomials involving only $x_1, \dots, x_k$ in the first k lines and of total degree $\le N$. Since $Dg(\mathbf{0})$ is invertible, g is invertible in some neighborhood of the origin in E^H.

Note that when $k = 1$ (which requires $|\lambda_1| > |\lambda_2|$), we can take $N = 1$, and then $g(\boldsymbol{x}) = \lambda_1 \boldsymbol{x}$.

Theorem 6. *There exists a neighborhood W of $\mathbf{0}$ in E^H such that the sequence of mappings $f^m \circ g^{-m} \colon W \to \mathbb{C}^n$ is defined for all $m \ge 0$, and converges uniformly to an analytic mapping $F \colon W \to \mathbb{C}^n$ whose derivative at $\mathbf{0}$ is the canonical injection $E^H \hookrightarrow \mathbb{C}^n$, and such that $f \circ F = F \circ g$.*

The image of F is the *k-very unstable manifold* of $\mathbf{0}$ for f. It is clearly invariant under f. The main difficulty in proving Theorem 6 is that it is not clear that $f^m \circ g^{-m}$ is defined on any neighborhood of $\mathbf{0}$ in E for all m, since f expands like multiplication by $|\lambda_1|$, whereas g^{-1} contracts like division by $|\lambda_k|$, so unless $k = 1$ it seems that f should win and that $f^m \circ g^{-m}(\boldsymbol{x})$ should be undefined for large m.

Throughout the paper, we will use the Euclidean norm on $\mathbb{C}^n$ denoted by $|\ |$; the norm on matrices is the operator norm with respect to the Euclidean norm.

Proof. Choose $\epsilon > 0$ so small that that $(|\lambda_k| - \epsilon)^N > |\lambda_1| + \epsilon$.

Make a linear change of variables so that $Df(\mathbf{0})$ satisfies $\|Df(\mathbf{0})\| < |\lambda_1| + \epsilon/2$. This is pure linear algebra: put $Df(\mathbf{0})$ into triangular form and scale the coordinates so that the off-diagonal terms are sufficiently small.

Next choose $\rho > 0$ so that

- f is defined on the ball $B_\rho(\mathbf{0})$ of radius ρ around $\mathbf{0}$,
- g is invertible on $B_\rho(\mathbf{0}) \cap E$,
- On $B_\rho(\mathbf{0})$ we have

$$\begin{aligned} &\|Df(\boldsymbol{x})\| \le |\lambda_1| + \epsilon \quad \text{so that} \quad |f(\boldsymbol{x})| \le (|\lambda_1| + \epsilon)|\boldsymbol{x}|, \\ &|g^{-1}(\boldsymbol{x})| \le \frac{|\boldsymbol{x}|}{|\lambda_k| - \epsilon} \quad \text{when} \quad \boldsymbol{x} \in E^H \cap B_\rho(\mathbf{0}), \\ &|f \circ g^{-1}(\boldsymbol{x}) - \boldsymbol{x}| \le C|\boldsymbol{x}|^N \quad \text{for some constant } C \end{aligned} \tag{4}$$

(this requires that f be prepared).

Define

$$\alpha = \frac{|\lambda_1| + \epsilon}{(|\lambda_k| - \epsilon)^N},$$

so that $0 < \alpha < 1$, and let r_0 be the positive solution of $r_0 + Cr_0^N/(1 - \alpha) = \rho$. Define r_m inductively by $r_{m+1} = r_m + C\alpha^m r_0^N$, so that $\lim_{m\to\infty} r_m = \rho$.

Theorem 6 now follows from Proposition 7, which is merely a more precise reformulation.

Proposition 7. (a) *For all $\boldsymbol{x} \in E^H$ with $|\boldsymbol{x}| < r_0$, all the compositions $f^m \circ g^{-m}(\boldsymbol{x})$ are defined.*

(b) *The sequence $f^m \circ g^{-m}$ converges uniformly on $E^H \cap B_{r_0}(\mathbf{0})$ to a map*

$$F \colon E^H \cap B_{r_0}(\mathbf{0}) \to B_\rho(\mathbf{0})$$

whose derivative is the canonical inclusion $E^H \subset \mathbb{C}^n$.

(c) *The map F satisfies the equation $f \circ F = F \circ g$.*

Proof. The hardest part is (a). We will prove by induction on m that if $|\boldsymbol{x}| < r_0$, then $f^l g^{-m}(\boldsymbol{x})$ is defined for all l with $0 \le l \le m$, and satisfies

$$|f^l \circ g^{-m}(\boldsymbol{x})| \le r_m \quad \text{for all } l \text{ satisfying } 0 \le l \le m.$$

This is certainly true for $m = 0$, so suppose it is true for some m and we will prove it for $m+1$.

First, let us see that the inductive hypothesis implies that for all l, $0 \le l \le m$, the map f^l is defined on the ball B of radius

$$C \frac{r_0^N}{(|\lambda_k| - \epsilon)^{mN}} \tag{5}$$

around $g^{-m}(\boldsymbol{x})$, and maps it into $B_{r_{m+1}}(\mathbf{0})$. If this were false, there would exist $l_0 \le m$ such that $f, f^2, \dots, f^{l_0-1}$ are all defined on B and bounded by r_{m+1} on B, and there exists $\boldsymbol{y} \in B$ such that $f^{l_0}(\boldsymbol{y})$ (which is defined since $|f^{l_0-1}(\boldsymbol{y})| < \rho$) satisfies $|f^{l_0}(\boldsymbol{y})| \ge r_{m+1}$. But this does not occur, since

$$\begin{aligned}
|f^{l_0}(\boldsymbol{y}) - f^{l_0} \circ g^{-m}(\boldsymbol{x})| &\le (|\lambda_1| + \epsilon)^{l_0} |\boldsymbol{y} - g^{-m}(\boldsymbol{x})| \\
&\le (|\lambda_1| + \epsilon)^{l_0} C (|\lambda_k| - \epsilon)^{mN} \\
&\le C \frac{(|\lambda_1| + \epsilon)^m}{(|\lambda_k| - \epsilon)^{mN}} |\boldsymbol{x}|^N = C\alpha^m |\boldsymbol{x}|^N < C\alpha^m r_0^N.
\end{aligned}$$

Thus

$$|f^{l_0}(\boldsymbol{y})| \le |f^{l_0}(\boldsymbol{y}) - f^{l_0}(g^{-m}(\boldsymbol{x}))| + |f^{l_0}(g^{-m}(\boldsymbol{x})| < C\alpha^m r_0^N + r_m = r_{m+1}.$$

With this under our belt, the rest is easy. We have

$$\begin{aligned}
|f^{m+1} \circ g^{-(m+1)}(\boldsymbol{x}) - f^m \circ g^{-m}(\boldsymbol{x})| &\le \big| f^m (f \circ g^{-1}(g^{-m}(\boldsymbol{x})) - f^m(g^{-m}(\boldsymbol{x}) \big| \\
&\le (|\lambda_1| + \epsilon)^m \big| (f \circ g^{-1}(g^{-m}(\boldsymbol{x})) - g^{-m}(\boldsymbol{x}) \big| \\
&\le (|\lambda_1| + \epsilon)^m C |g^{-m}(\boldsymbol{x})|^N \\
&\le C \frac{(|\lambda_1| + \epsilon)^m}{(|\lambda_k| - \epsilon)^{mN}} |\boldsymbol{x}|^N = C\alpha^m |\boldsymbol{x}|^N,
\end{aligned}$$

where the term $f^{m+1} \circ g^{-(m+1)}(\boldsymbol{x})$ is well defined because it can be rewritten $f^m(f \circ g^{-1}(g^{-m}(\boldsymbol{x})))$, and $f \circ g^{-1}(g^{-m}(\boldsymbol{x}))$ belongs to the ball B given in Equation (5). We have also used the chain rule, which requires that all the intermediate iterations belong to $B_\rho(\mathbf{0})$, which was also justified above.

This proves (a), but it also proves (b), since it shows that the series on the right of

$$\lim_{m \to \infty} f^m \circ g^{-m}(\boldsymbol{x}) = \boldsymbol{x} + \big(f \circ g^{-1}(\boldsymbol{x}) - \boldsymbol{x}\big) + \big(f^2 \circ g^{-2}(\boldsymbol{x}) - f \circ g^{-1}(\boldsymbol{x})\big) + \dots$$

converges uniformly and absolutely on the ball of radius r_0. The series can be differentiated term by term, and the only term with non-vanishing derivative at the origin is the first, giving the canonical inclusion $E^H \to \mathbb{C}^n$.

Finally, (c) follows from

$$F(g(\boldsymbol{x})) = \lim_{m \to \infty} f^m \circ g^{-m+1}(\boldsymbol{x}) = f \circ \lim_{m \to \infty} f^{m-1} \circ g^{-(m-1)}(\boldsymbol{x}) = f(F(\boldsymbol{x})).$$

This completes the proof of Proposition 7 and Theorem 6. □

5. Local Existence of Very Stable Manifolds

In the large, stable manifolds behave quite differently from unstable manifolds when f is not an automorphism. But locally, if f is locally invertible, it is easy to adapt the proof for the unstable manifolds to show existence for stable manifolds.

Let $U \subset \mathbb{C}^n$ be a neighborhood of $\mathbf{0}$ and $f: U \to \mathbb{C}^n$ be an analytic mapping such that $f(\mathbf{0}) = \mathbf{0}$ and $Df(\mathbf{0})$ has eigenvalues $\lambda_1, \ldots, \lambda_n$ satisfying

$$0 < |\lambda_1| \leq \cdots \leq |\lambda_k| < |\lambda_{k+1}| \leq \cdots \leq |\lambda_n| \quad \text{and} \quad |\lambda_k| < 1.$$

Suppose that N is an integer such that $|\lambda_k|^N < |\lambda_1|$, and suppose that f^{-1} is (N, k)-prepared. Let E^L be the generalized eigenspace corresponding to the first k eigenvalues, and EH the direct sum of the generalized eigenspaces corresponding to the other eigenvalues.

Corollary 8. *Let*

$$g = \mathrm{pr}^L \circ (f|_{E^L}) : E^L \to E^L$$

be the polynomial map obtained by truncating the Taylor expansion of f to degree N, restricting to E^L and projecting on E^L. Then there exists a neighborhood $W \subset E^L$ of $\mathbf{0}$ and an analytic mapping $F: W \to \mathbb{C}^n$ mapping $\mathbf{0}$ to $\mathbf{0}$, such that $DF(\mathbf{0})$ is the canonical inclusion $E^L \hookrightarrow \mathbb{C}^n$, and such that $F \circ g = f \circ F$.

Proof. Since $\lambda_1 \neq 0$, the mapping f is locally invertible, and we simply apply Theorem 6 to f^{-1}. □

The image of F will be called the *k-very stable manifold* of $\mathbf{0}$.

6. Analytic Dependence of Very Unstable Manifolds on Parameters

Let $t_0 \in T$ be an analytic manifold with a distinguished point. Let $U \subset \mathbb{C}^n$ be a neighborhood of $\mathbf{0}$, and let $f_t: U \to \mathbb{C}^n$, $t \in T$, be an analytic family of analytic mappings parametrized by T. Suppose that for all t we have $f_t(\mathbf{0}) = \mathbf{0}$ and that f_{t_0} satisfies the conditions of Theorem 6.

We want our parametrization of the k-very unstable manifolds for f_t to depend analytically on the parameter t, and there is a problem. Our construction of F depends in an essential way on preparing f, and our proof that f can be prepared depends on the matrix of $Df(\mathbf{0})$ being triangular. Of course nothing stops us from taking $Df_{t_0}(\mathbf{0})$ triangular, but in general we cannot find a linear change of variables that depends analytically on t and makes all $Df_t(\mathbf{0})$ triangular. If there were such a thing, then the entries on the diagonal of the $Df_t(\mathbf{0})$, i.e., the eigenvalues, would also depend analytically on t, and the roots of a polynomial do not depend analytically on the coefficients when the polynomial has multiple roots.

Of course, we could simply take as a hypothesis that all $Df_t(\mathbf{0})$ are triangular, or that they can be analytically triangularized. But that would be a serious weakening of our desired result, as we would usually lose the analytic dependence whenever $Df_{t_0}(\mathbf{0})$ has multiple eigenvalues.

First, let us give an example to illustrate the difficulties which arise.

Example 9. Let

$$f_t \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} \lambda^3 x_1 + x \\ t x_1 + \lambda^3 x_2 \\ \lambda x_3 + x_4 + b_1 x_1 x_2 \\ t x_3 + \lambda x_4 + b_2 x_1 x_2 \end{pmatrix},$$

where we assume $|\lambda| > 1$; we will attempt to construct the 2-very unstable manifold.

The matrix $\begin{bmatrix} \lambda & 1 \\ t & \lambda \end{bmatrix}$ cannot be triangularized analytically in t, since the eigenvalues are $\lambda \pm \sqrt{t}$ do not depend analytically on t; neither can $\begin{bmatrix} \lambda^3 & 1 \\ t & \lambda^3 \end{bmatrix}$. Thus the linear terms beneath the diagonal cannot be eliminated by a change of variables that depends analytically on t.

In this case to prepare f_t we must eliminate (3, 2)-offending terms. The first such term is the monomial $b_2 x_1 x_2$ in the fourth term. If we conjugate by $\Phi^4_{[1,1,0,0]}(a)$ as in the proof of Proposition 4, the fourth coordinate is

$$t x_3 + \lambda x_4 + x_1 x_2 (a(\lambda - \lambda^6) + b_2) - a t \lambda^3 x_1^2 - a \lambda^3 x_2^2.$$

As in Proposition 4, we can choose a so as to eliminate the offending term, but at the cost of introducing the new offending term $-a t \lambda^3 x_1^2$, which in our order comes *before* the offending term $p^4_{[1,1,0,0]}$ just eliminated. Even if we ignore this, a similar computation will show that when we eliminate the term $b_1 x_1 x_2$ in the third coordinate, we will introduce a term in $x_1 x_2$ back into the fourth coordinate. Eliminating offending terms one at a time is hopeless.

Proposition 10. *Let the eigenvalues of $Df_{t_0}(\mathbf{0})$ satisfy the conditions (3), suppose N is chosen so that $|\lambda_k|^N > |\lambda_1|$. Then there exists a neighborhood $T' \subset T$ of t_0 and a polynomial change of variables Φ_t depending analytically on $t \in T'$ such that $(\Phi_t)^{-1} \circ f_t \circ \Phi_t$ is (N, k) prepared.*

Proof. Let the distinct eigenvalues of $Df_{t_0}(\mathbf{0})$ be $\mu_1, \ldots, \mu_m$ with multiplicities $l_1, \ldots, l_m$; and suppose that $|\mu_1| \geq \cdots \geq |\mu_m|$.

Lemma 11. *There is a neighborhood $T' \subset T$ of t_0 and a linear change of variables Φ_t depending analytically on $t \in T'$ so that if we set $g_t = \Phi_t^{-1} \circ f_t \circ \Phi_t$, then the matrices of $Dg_t(\mathbf{0})$ are all block diagonal, i. e., all entries outside of square blocks of side $l_1, \ldots, l_m$ along the diagonal vanish, and the matrix $Dg_{t_0}(\mathbf{0})$ is upper triangular.*

Note that this already says that f_t has no *linear* offending terms for $t \in T'$.

Proof of Lemma 11. One way of approaching this is via complex analysis. Choose discs D_i around the μ_i, and let T' be so small that the spectrum of $Df_t(\mathbf{0})$ is contained in these discs for all $t \in T'$. Then the operators $P_i(t)\colon \mathbb{C}^n \to \mathbb{C}^n$ given by

$$P_i(t) = \frac{1}{2\pi i} \int_{\partial D_i} (\zeta - Df_t(\mathbf{0}))^{-1} d\zeta$$

are projectors onto the direct sum of the generalized eigenspaces corresponding to the eigenvalues of $Df_t(\mathbf{0})$ in D_i. These projectors evidently vary analytically with

t for $t \in T'$, and it is easy to see that the rank of each P_i is l_i for all $t \in T'$. Denote by $E_1(t), \dots, E_m(t)$ the images of $P_1(t), \dots, P_m(t)$. Choose bases $\boldsymbol{v}_{i,1}, \dots, \boldsymbol{v}_{i,l_i}$ of $E_i(t_0)$ such that in these bases $Df_{t_0}(\mathbf{0})$ is upper triangular, for instance by taking $Df_{t_0}(\mathbf{0})$ in upper Jordan canonical form. Then set

$$\boldsymbol{v}_{i,j}(t) = P_i(t)(\boldsymbol{v}_{i,j})$$

These $\boldsymbol{v}_{i,j}(t)$ will form a basis of $\mathbb{C}^n$ for t in some perhaps smaller neighborhood of t_0, and in this basis the matrix of $Df_t(\mathbf{0})$ has the desired block-diagonal form, since the $E_i(t)$ are invariant under $Df_t(\mathbf{0})$. □

Now we will deal with all offending terms of each degree $2, \dots, N$ at once rather than one at a time. By induction, suppose that for some $q \le N$, the maps f_t have no offending terms of degree $< q$. We will conjugate f_t by Φ_t given by

$$\Phi_t(\boldsymbol{x}) = \boldsymbol{x} + \sum_{p_I^j \text{ offending, } |I|=q} a_I^j(t) p_I^j(\boldsymbol{x}).$$

One way of understanding the proof of Proposition 4 is to say that the numbers a_I^j for which the mapping

$$(\Phi_{t_0})^{-1} \circ f_{t_0} \circ \Phi_{t_0}$$

has no offending terms of degree q are the solutions of a system of linear equations such that if we order the unknowns according to Equation (2), then the matrix L_{t_0} of coefficients is upper triangular, with entry $\lambda_j - \boldsymbol{\lambda}^I$ on the diagonal in the column corresponding to the unknown $a_I^j(t_0)$. In particular this matrix is invertible.

Now the assertion that

$$(\Phi_t)^{-1} \circ f_t \circ \Phi_t$$

has no offending terms of degree q is also a system of linear equations, for the same unknowns. This time the matrix L_t is not triangular; that is why we cannot eliminate the offending terms one at a time. But it is of the form

$$L_t = L_{t_0} + (t - t_0) M_t;$$

in particular, it is still invertible for t sufficiently close to t_0. Thus for t sufficiently close to t_0 we can choose analytic functions $a_I^j(t)$ so that after conjugating by Φ_t the mapping f_t has no offending monomials of degree q. It is easy to see that we have introduced no new offending terms of degree $< q$. □

With this preparation, Theorem 12 is easy, and its proof is left to the reader.

Theorem 12. *Suppose that for all t we have $f_t(\mathbf{0}) = \mathbf{0}$, that f_{t_0} satisfies the conditions of Theorem 6. Then the hypotheses of Theorem 6 are satisfied for t in some neighborhood $T' \subset T$ of t_0, and if we denote by g_t, F_t the mappings constructed in Theorem 6 from f_t, then F_t depends analytically on t in T'.*

Remark 13. Note the hypothesis that $f_t(\mathbf{0}) = \mathbf{0}$. In many settings, we have a family of analytic maps f_t, but no point that is fixed by all f_t. Corollary 12 applies anyway in a neighborhood of a fixed point $\boldsymbol{p}$ of f_{t_0}, so long as none of the eigenvalues of $Df_{t_0}(\boldsymbol{p})$ is 1. Under this hypothesis, the implicit function theorem implies that there is locally near t_0 an analytic map $t \mapsto \boldsymbol{p}(t)$ with $\boldsymbol{p}(t_0) = \boldsymbol{p}$ and $f_t(\boldsymbol{p}(t)) = \boldsymbol{p}(t)$

for all t in a neighborhood of t_0. But if 1 is an eigenvalue of $Df_{t_0}(\boldsymbol{p})$, then usually the fixed point $\boldsymbol{p}$ bifurcates, and Corollary 12 does not apply.

7. The Naturality of F

There are two questions involved in the naturality of very unstable manifolds: is the *image* of F unique, and is the parametrization natural. We will see in this section that the answer to the first is an unqualified yes, and in the next that the answer to the second is a qualified no.

Again, suppose that U is a neighborhood of 0 in $\mathbb{C}^n$, that $f\colon U \to \mathbb{C}^n$ is an analytic mapping with $f(\mathbf{0}) = \mathbf{0}$, and that the eigenvalues of $Df(\mathbf{0})$ are $\lambda_1, \ldots, \lambda_n$, ordered so that

$$|\lambda_1| \geq \cdots \geq |\lambda_k| > |\lambda_{k+1}| \geq \cdots \geq |\lambda_n|.$$

Recall that E^H and E^L are the direct sums of the generalized eigenspaces corresponding to the high and the low eigenvalues.

Then a k-very unstable manifold will be the graph of an analytic mapping $\alpha\colon W \to E^L$ without constant or linear terms, and defined on some neighborhood $W \subset E^H$ of $\mathbf{0}$. Moreover, the invariance of the manifold is expressed by the following equation:

$$f\begin{pmatrix}\boldsymbol{x}\\ \alpha(\boldsymbol{x})\end{pmatrix}_L = \alpha(f\begin{pmatrix}\boldsymbol{x}\\ \alpha(\boldsymbol{x})\end{pmatrix}_H), \tag{6}$$

where the indices indicate projection onto E^L and E^H respectively.

We already know that the image of F is the graph of such a map α, so the following proposition settles the existence and uniqueness problem for the very unstable manifold, at least in the analytic category.

Proposition 14. *Equation (6) has a unique solution in the ring of formal power series.*

Proof. Choose an (H, L)-basis for which $Df(\mathbf{0})$ is in upper-triangular form, and write

$$Df(\mathbf{0}) = \begin{bmatrix}\lambda_{1,1} & \lambda_{1,2} & \ldots & \lambda_{1,k}\\ 0 & \lambda_{2,2} & \ldots & \lambda_{2,k}\\ \ldots & \ldots & \ldots & \ldots\\ 0 & 0 & \ldots & \lambda_{k,k}\end{bmatrix},$$

where $\lambda_i = \lambda_{i,i}$.

Let the series for α be $\alpha = \sum_{|I|\geq 2}\sum_{j=k+1}^{n} a_I^j p_I^j$, where the multiindex I only concerns the first k variables. Substitute this expression in Equation (6). Clearly the lowest degree term in which a_I^j appears is contributed by the linear terms of f. These give the equation

$$a_I^j(\lambda_{1,1}x_1 + \cdots + \lambda_{1,k}x_k)^{i_1}(\lambda_{2,2}x_2 + \cdots + \lambda_{2,k}x_k)^{i_2}\cdots(\lambda_{k,k}x_k)^{i_k} = a_I^j\lambda_j\boldsymbol{x}^I. \tag{7}$$

In particular, the coefficient of $\boldsymbol{x}^I$ is $\lambda_1^{i_1}\cdots\lambda_k^{i_k} - \lambda_j$, and since all $|\lambda_i| > |\lambda_j|$ for $1 \leq i \leq k$, we see that this coefficient does not vanish. Since all the monomials of degree $|I|$ that appear in Equation (7) come later in the ordering, the coefficients can be recursively computed. □

It is essential that the eigenvalues we are considering be the big ones. There is not necessarily any invariant manifold corresponding to small eigenvalues, and if one does exist, it may fail to be unique. Thus even though we are considering a formal power series computation, the magnitudes of the eigenvalues have to play a role.

Example 15. Consider the map

$$f\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}4x\\2y\end{pmatrix}.$$

Then the curves of equation $x = Cy^2$ are all invariant, and all tangent to the y-axis, which is the eigenspace for the small eigenvalue 2.

If instead we consider

$$f\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}4x+y^2\\2y\end{pmatrix},$$

then there is no invariant analytic curve tangent to the y-axis. Indeed, if we set $x = \alpha(y) = a_2y^2 + a_3y^3 + \dots$, then Equation (6) leads to $4a_2 = 4a_2 + 1$.

8. The Affine Structure of Very Unstable Manifolds and Resonant Forms

We have proved that on its very unstable manifold, a mapping f is locally conjugate to a certain polynomial map g, obtained by truncating the Taylor series of f, restricting and projecting. The mapping g is polynomial, hence defined on all of $\mathbb{C}^n$, but its global dynamics may be complicated: it will probably have critical points, and even if it doesn't, it may be as complicated as a Hénon mapping.

Of course, the restriction of f to the very unstable manifold is also locally conjugate to any other map locally conjugate to g, and we may wonder whether there is a nicer one.

The obvious candidate is the restriction of $Df(\mathbf{0})$ to E^H, and usually (as we will see), g is conjugate to this linear map. Even when it isn't, there is a fairly nice map conjugate to g, called a *resonant form*, which reduces to a linear map when there are no "resonances". This resonant form is already interesting locally: it almost provides a normal form for g. But it is much more interesting globally; although g might have complicated global dynamics, resonant forms don't: in our setting, they are bijective and all points iterate to ∞ under g and to $\mathbf{0}$ under g^{-1}.

What do we mean by "globally"? Suppose X is a complex manifold (often $\mathbb{C}^n$ in practice), and $f\colon X \to X$ is analytic, so that in particular all iterates f^m are defined on all of X. Suppose $\boldsymbol{p} \in X$ is a fixed point of f, and that $Df(\boldsymbol{p})$ satisfies the conditions of Theorem 6. If M is the local very unstable manifold provided by Theorem 6, we can consider the global unstable manifold given by the increasing union

$$\widetilde{M} = \bigcup_{m=0}^{\infty} f^m(M).$$

What is the structure of $\widetilde{M}$? As a subset of X, it tends to be terribly complicated, usually dense in some complicated fractal invariant subset of X. (Understanding the structure of this fractal is what one means by "understanding f", something that has seldom been achieved for any interesting class of maps in dimension greater than 1.) But the intrinsic structure of $\widetilde{M}$ is never complicated: there is always a holomorphic mapping $F\colon \mathbb{C}^k \to \widetilde{M}$ such that $f \circ F = F \circ g$ for some resonant form g. Thus we think of $\widetilde{M}$ as all tangled up in X, like a sheet all tangled in a washing machine, and of F as "untangling" $\widetilde{M}$ into $\mathbb{C}^k$ (with a resonant form $\mathbb{C}^k \to \mathbb{C}^k$), like the same sheet spread out to dry.

We begin by defining resonant monomials and resonant forms. These depend on a linear map $L\colon \mathbb{C}^k \to \mathbb{C}^k$, whose matrix we will assume to be upper triangular, with eigenvalues $\lambda_1, \ldots, \lambda_k$ satisfying $|\lambda_1| \geq \cdots \geq |\lambda_k|$. The monomial mapping p_I^j is called *resonant* with respect to L if

$$\lambda_j = \lambda_1^{i_1} \cdot \cdots \cdot \lambda_k^{i_k} = \boldsymbol{\lambda}^I. \tag{8}$$

A mapping $f\colon \mathbb{C}^k \to \mathbb{C}^k$ is called a *resonant form* with respect to L if all its monomials are resonant.

Remark. We are allowing $|I| = 1$; this means that p_i^j is resonant if $\lambda_i = \lambda_j$. Thus the off-diagonal terms of Jordan canonical form are resonant; and indeed it is well known that they cannot be eliminated by conjugation. We have already encountered condition (8) several times, first in Proposition 2 and most recently in Equation (7).

Proposition 16. *Let $L\colon \mathbb{C}^k \to \mathbb{C}^k$ be as above, upper triangular with eigenvalues satisfying $|\lambda_1| \geq \cdots \geq |\lambda_k| > 1$ and suppose $|\lambda_k|^N > |\lambda_1|$. Then any resonant form $g\colon \mathbb{C}^k \to \mathbb{C}^k$ with $g(\mathbf{0}) = \mathbf{0}$ and $Dg(\mathbf{0}) = L$ has degree $< N$ and is invertible*; *and under iteration of g all points $\boldsymbol{x} \neq 0$ iterate to infinity, and they all iterate to $\mathbf{0}$ under g^{-1}.*

Proof. Since for any I we have

$$|\boldsymbol{\lambda}^I| \geq |\lambda_k|^{|I|} > |\lambda_1|,$$

no monomial resonant with respect to L can have degree $\geq N$.

Our condition implies that the kth line of the equation $g(\boldsymbol{y}) = \boldsymbol{x}$ is simply $\lambda_k y_k = x_k$, since

$$\lambda_k = \lambda_1^{i_1} \cdot \cdots \cdot \lambda_k^{i_k}$$

implies that $i_1 = \cdots = i_{k-1} = 0$ and $i_k = 1$; now use the fact that $Dg(\mathbf{0})$ is triangular. Thus we can solve for y_k in terms of x_k.

More generally, the jth line of the equation $g(\boldsymbol{y}) = \boldsymbol{x}$ is of the form

$$\lambda_j y_j + (\text{terms involving only } y_{j+1}, \ldots, y_k) = x_j.$$

Indeed, the monomial p_I^j is resonant if $\lambda_j = \boldsymbol{\lambda}^I$, and since we have $|\lambda_i| > 1$ for all i, only λ_l with $l \geq j$ can occur in I, except for linear terms for $j' > j$ with $\lambda_{j'} = \lambda_j$; the triangular form of $Dg(\mathbf{0})$ says that this also occurs only if $j' \geq j$. Thus if we have recursively solved for $y_{j+1}, \ldots, y_k$ in terms of $x_{j+1}, \ldots, x_k$, we can solve for $y_j, \ldots, y_k$ in terms of $x_j, \ldots, x_k$.

Continue until you get to y_1.

This proves that g is invertible. If a point $\boldsymbol{x} \in \mathbb{C}^k$ does not iterate to infinity under g, then the formula $\lambda_k y_k = x_k$ shows that $x_k = 0$. But then $x_{k-1} = 0$ also, and so forth until we get to x_1. The same argument shows that all points iterate to 0 under g^{-1}. □

When we try to extend this result to an analytic family of mappings f_t, we run into the same difficulties that we encountered in Proposition 10: the linear terms of the perturbed mapping cannot always be made diagonal, and the terms beneath the diagonal prevent us from solving for the coordinates of $f_t(\boldsymbol{y}) = \boldsymbol{x}$ one at a time.

Proposition 17. *Let $L\colon \mathbb{C}^k \to \mathbb{C}^k$ be as above, upper triangular with eigenvalues satisfying $|\lambda_1| \geq \cdots \geq |\lambda_k| > 1$. and suppose $|\lambda_k|^N > |\lambda_1|$. Let $t_0 \in T$ be a distinguished point of an analytic manifold, and $g_t\colon \mathbb{C}^k \to \mathbb{C}^k$ be an analytic family of polynomial maps, such that g_{t_0} is a resonant form for L, and such that all g_t have non-zero coefficients only for monomials that are resonant for L. Then there exists a neighborhood T' of t_0 such that all g_t, $t \in T'$, are invertible and have the simple dynamics of resonant forms: all points $\boldsymbol{x} \neq \mathbf{0}$ iterate to infinity under g, and they all iterate to $\mathbf{0}$ under g^{-1}.*

Remark. We are not assuming that $Dg_t(\mathbf{0}) = L$ for $t \neq t_0$, so in all likelihood the monomials of g_t are not resonant for g_t. But we cannot eliminate them if we want holomorphic dependence.

Proof. We need to modify the proof of Proposition 16. The difficulty comes from the linear terms: the matrix $Dg_t(\mathbf{0})$ may acquire terms below the diagonal (in positions (i, j) where $\lambda_i = \lambda_j$; we have seen that these are resonant). This prevents us from solving the equation $g_t(\boldsymbol{y}) = \boldsymbol{x}$ for the variables $y_k, \ldots, y_1$ one at a time.

We will overcome this as in Proposition 10 by considering all the variables corresponding to each generalized eigenspace at once. Let $\mu_1, \ldots, \mu_m$ be the distinct eigenvalues of $Dg(\mathbf{0})$, and E_{μ_i} be the generalized eigenspace for μ_i. Let us denote by $\boldsymbol{x}_\mu$, $\boldsymbol{y}_\mu$ the variables of the eigenspace E_μ. Then the equation $g_t(\boldsymbol{y}) = \boldsymbol{x}$ for the vector $\boldsymbol{y}_\mu$ in that eigenspace becomes

$$(Dg_t(\mathbf{0})|_{E_\mu})(\boldsymbol{y}_\mu) + (\text{terms involving only previously computed } y\text{'s}) = \boldsymbol{x}_\mu.$$

This can be solved because $Dg_t(\mathbf{0})|_{E_\mu}$ is a small perturbation of the invertible matrix $Dg_{t_0}(\mathbf{0})|_{E_\mu}$. □

As in Theorem 6, let us suppose that U is a neighborhood of $\mathbf{0}$ in $\mathbb{C}^n$, and that $f\colon U \to \mathbb{C}^n$ is an analytic mapping with $f(\mathbf{0}) = \mathbf{0}$. Again we suppose that the eigenvalues of $Df(\mathbf{0})$ satisfy the conditions (3), and E^H, E^L have the same meaning as in Section 3, and choose an (H, L)-basis with respect to which $Df(\mathbf{0})$ is upper triangular. Choose an integer N such that $|\lambda_k|^N > |\lambda_1|$.

The restriction $L = Df(\mathbf{0})|E^H$ is the linear map underlying our resonances, and although we are considering maps $f\colon \mathbb{C}^n \to \mathbb{C}^n$, we will be particularly interested in monomials involving only the first k variables, which we will call *k-monomials* (i. e., monomial mappings p_I^j where $j \leq k$ and $I \in \mathbb{N}^n$ satisfies $i_{k+1} = \cdots = i_n = 0$).

We next show that our map f can be conjugated so that its k-monomials of degree $< N$ are all resonant.

Proposition 18. *There exists a neighborhood $V \subset \mathbb{C}^n$ of $\mathbf{0}$ and a mapping $\Phi\colon V \to \mathbb{C}^n$ tangent to the identity such that if we set $f_\Phi = \Phi^{-1} \circ f \circ \Phi$ and define*

$$g = \mathrm{pr}^H \circ (f_\Phi|_{E^H})\colon E^H \to E^H,$$

then g is a resonant form.

Proof. This follows immediately from Proposition 2, which says that we can get rid of non-resonant terms one at a time in the order (2), without introducing others which come earlier in the order. We might as well get rid of all non-resonant monomials of degree $\leq N$; there will then be no non-resonant monomials in the first k coordinates involving only the variables of E^H. $\square$

Remark. Proposition 18 is simple because of the hypothesis $|\lambda_k| > 1$. If there are eigenvalues λ_j with $|\lambda_j| < 1$, there may be infinitely resonant monomials. More serious, there usually will be infinitely many "near resonances", and these lead to very interesting but difficult small divisor problems. But these will involve both the variables of E^H and of E^L, and need not concern us here.

The following example illustrates this computation.

Example 19. Let

$$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \lambda^3 x + yz \\ \lambda y + z \\ \lambda z + byz \end{pmatrix}, \quad |\lambda| > 1.$$

The term yz in the third coordinate is not resonant, so we can eliminate it. Set

$$\Phi\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z + ayz \end{pmatrix}, \quad \text{so that} \quad \Phi^{-1}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z - ayz + a^2y^2z - \dots \end{pmatrix}.$$

Here, as is typical, we have

$$\Phi^{-1} = I - ap_I^j + (\text{terms of degree} \geq 2|I| - 1)$$

and the extra terms do not affect our conclusions (you have to think carefully about the linear terms: if $|I| = 1$, then $2|I| - 1 = |I|$). Now compute the conjugation (omitting most terms of degree > 2):

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \overset{\Phi}{\mapsto} \begin{pmatrix} x \\ y \\ z + ayz \end{pmatrix} \overset{f}{\mapsto} \begin{pmatrix} \lambda^3 x + y(z + ayz) \\ \lambda y + z + ayz \\ \lambda(z + ayz) + by(z + ayz) \end{pmatrix} \overset{\Phi^{-1}}{\mapsto}$$
$$\begin{pmatrix} \lambda^3 x + yz + ay^2z \\ \lambda y + z + ayz \\ \lambda z + yz(\lambda a + b) - a(\lambda y + z)\lambda z \end{pmatrix}.$$

The term that really matters is $yz(b + a(\lambda - \lambda^2))$ in the third coordinate, which can be made to vanish by setting $a = -b/(\lambda - \lambda^2)$. This reflects the fact that the polynomial yz in the third entry, i. e., $p^3_{(0,1,1)}$, is not resonant. The other new terms of degree 2 are

(1) the term ayz in the second term. This is later than the term yz in the third term because of the third condition in the definition of $\prec$, reflecting the fact that the linear terms of f are upper triangular, and
(2) the term $-a\lambda z^2$ in the third coordinate; this is also later by the second condition. Again this reflects the fact that L is upper triangular; we are substituting coordinate functions with higher indices in p_I^j.

There are lots of new terms of degree 3 or higher, which will be dealt with later. Note the term ay^2z in the first line: it is resonant, even though there was no such resonant term in the original mapping.

Resonances occur on a closed set, so one would not expect the resonant form g of a mapping $f\colon E \to E$ to depend analytically on f. But we can do almost as well; we can associate to a family of mappings f_t a family of mappings g_t, so that g_{t_0} is resonant, but all g_t have the same simple dynamics as resonant mappings.

Corollary 20. *Let $t_0 \in T$ be an analytic manifold with a distinguished point. Let $U \subset \mathbb{C}^n$ be a neighborhood of $\mathbf{0}$, and $f_t\colon U \to \mathbb{C}^n$, $t \in T$, be an analytic family of analytic mappings parametrized by T, such that $f_t(\mathbf{0}) = \mathbf{0}$ for all $t \in T$. Moreover, suppose that f_{t_0} satisfies the conditions of Theorem 6. Then there exists a neighborhood T' of t_0 and an analytic family of locally invertible mappings Φ_t, $t \in T'$, such that the non-zero k-monomials of*

$$\Phi_t \circ f_t \circ \Phi_t^{-1}$$

of degree $< N$ are all resonant monomials for $Df_{t_0}(\mathbf{0})$.

Proof. We have encountered the difficulty in this proof twice already: the possible terms beneath the diagonal of $Df_t(\mathbf{0})$ prevent us from eliminating terms one by one. We have also solved the problem twice, and we will use the same method here. The equation for eliminating from the power series of f_t all monomials of a given degree that are non-resonant for Df_{t_0} leads to a system of linear equations for the coefficients of the conjugating map. This system is a small perturbation of the system for f_{t_0}, and for that system, Proposition 18 tells us that the matrix of coefficients is upper triangular with non-zero entries on the diagonal.

Note that we are carefully not trying to eliminate the terms of $Df_t(\mathbf{0})$ that might be below the diagonal, and that correspond to resonant linear terms of Df_{t_0}.

Now we can claim that a small perturbation of an invertible matrix is invertible, so the system can be solved. □

9. The Global Structure of Unstable Manifolds

We now come to our main result on the global structure of unstable manifolds. Let X be a complex manifold, $f\colon X \to X$ be an analytic mapping, and $\boldsymbol{p} \in X$ be a fixed point of f. Suppose the eigenvalues of $Df(\boldsymbol{p})$ are $\lambda_1, \dots, \lambda_n$, ordered so that

$$|\lambda_1| \geq \cdots \geq |\lambda_k| > |\lambda_{k+1}| \geq \cdots \geq |\lambda_n|,$$

and that $|\lambda_k| > 1$.

We know that there is a local very unstable manifold M in some neighborhood of $\boldsymbol{p}$, and we can consider the global object $\widetilde{M} = \bigcup_{m\geq 0} f^m(M)$. Theorem 21 tells us how to parametrize $\widetilde{M}$.

Theorem 21. *There exists a neighborhood U of $\mathbf{0} \in T_{\boldsymbol{p}}X$, a local chart $\phi\colon U \to X$ tangent to the identity, and a resonant form $g\colon E^H \to E^H$ such that the limit*

$$F = \lim_{m\to\infty} f^m \circ \phi \circ g^{-m}$$

exists for all $\boldsymbol{x} \in E^H$. This limit defines an mapping $F\colon E^H \to X$ whose image is $\widetilde{M}$, and such that $f \circ F = F \circ g$.

Proof. This is simply a matter of putting together Theorem 6 and Propositions 18 and 16.

By Proposition 18, there exists a neighborhood U of $\mathbf{0} \in T_{\boldsymbol{p}}X$ and local coordinate $\phi\colon U \to X$ tangent to the identity such that $\phi^{-1} \circ f \circ \phi$ satisfies the conclusion of Proposition 18. This gives us a resonant form $g\colon E^H \to E^H$, and applying Proposition 16, we know that for any compact set $K \subset E^H$, there exists q such that $g^q(K) \subset U$. Now on K we can write

$$F = \lim_{m\to\infty} f^m \circ \phi \circ g^{-m} = f^q \circ \lim_{m\to\infty} \left(f^{m-q} \circ \phi \circ g^{-(m-q)}\right) \circ g^q,$$

and by Theorem 6, the limit exists, and satisfies $f \circ F = F \circ g$. Being a limit of analytic mappings that converges uniformly on compact sets, it is analytic. The image of F contains the local k-very unstable manifold M of $\boldsymbol{p}$, hence $\widetilde{M} = \bigcup_i f^i(M)$. □

Now we put analytic parameters in the construction. Let $t_0 \in T$ be an analytic manifold with a distinguished point, and $f_t\colon X \to X$, $t \in T$, be an analytic family of analytic mappings parametrized by T. Suppose that $\boldsymbol{p}(t)$ is a point of X depending analytically on $t \in T$, such that for all t we have $f_t(\boldsymbol{p}(t)) = \boldsymbol{p}(t)$, and such that f_{t_0} satisfies the hypotheses of Theorem 21 with respect to $\boldsymbol{p} = \boldsymbol{p}(t_0)$. We can then consider the "eigenspace" E_t for $t \in T'$ for an appropriate neighborhood T' of T_0, and we will choose a basis $\boldsymbol{e}_1(t), \dots, \boldsymbol{e}_k(t)$ of E_t, depending analytically on $t \in T'$. Our construction will depend on this choice of basis.

Corollary 22. *There exists a family $g_t\colon \mathbb{C}^k \to \mathbb{C}^k$ of maps that depend analytically on t in a neighborhood T' of t_0, all having the simple dynamics of resonant forms, and a family ϕ_t of local coordinates near $\boldsymbol{p}(t)$ such that the limit*

$$F_t = \lim_{m\to\infty} f_t^m \circ \phi_t \circ g_t^{-m}$$

exists uniformly on compact subsets of $T' \times \mathbb{C}^k$, and gives for each $t \in T'$ a parametrization of the unstable manifold of $\boldsymbol{p}(t)$ in the sense of Theorem 21.

Proof. Use Proposition 20 to choose local coordinates near the $\boldsymbol{p}(t)$ so that the expression of f_t has only appropriate monomials, then use Proposition 17 to assert that the mappings g_t constructed have the desired simple dynamics, and finally use Theorem 6 to guarantee the convergence, as in the proof of Theorem 21. □

10. The Critical Points of F

In general F is neither injective, nor an immersion.

Example 23. Consider the mapping $f(z) = z^2 - 2$, with the fixed point 2, where $f'(2) = 4$. In that case, the map $F(t) = 2\cos\sqrt{t}$ satisfies

$$f(F(t)) = F(4t),$$

and it is in fact the limit

$$F(t) = \lim_{m\to\infty} f^m\left(2 + \frac{t}{4^m}\right).$$

But it is clearly neither injective nor an immersion.

It is easy to see what the critical points of F are.

Proposition 24. *Suppose $f\colon X \to X$ and $\boldsymbol{p}$ satisfy the hypotheses of Theorem 21. Let $Z \subset X$ be the critical locus of f. The critical points of F are a subset of*

$$\bigcup_{m>0} g^m(F(\mathbb{C}^k) \cap Z).$$

Proof. Differentiate the equation $f^m \circ F = F \circ g^m$, to find

$$\big(D(f^m)(F(\boldsymbol{x}))\big)DF(\boldsymbol{x}) = \big(DF(g^m(\boldsymbol{x}))\big)D(g^m)(\boldsymbol{x}).$$

We see that the only way to have $DF(g^m(\boldsymbol{x}))$ non-injective is for either $DF(\boldsymbol{x})$ or $D(f^m)(F(\boldsymbol{x}))$ to fail to be injective. But any point $\boldsymbol{x} \in E^H$ can be written $g^m(\boldsymbol{y})$ for some $\boldsymbol{y}$ such that $DF(\boldsymbol{y})$ is certainly injective. Thus $DF((g^m)(\boldsymbol{y}))$ non-injective implies that $D(f^m)(F(\boldsymbol{y}))$ also non-injective, which means that the orbit of $\boldsymbol{y}$ passes through Z. □

If $f\colon X \to X$ is an automorphism (or more generally an injective immersion), this leads to the following result.

Corollary 25. *Let $f\colon X \to X$ be an automorphism, and $\boldsymbol{p}$ a fixed point of f satisfying the hypotheses of Theorem 21. Then the parametrization $F\colon \mathbb{C}^k \to X$ of the k-very unstable manifold constructed in that theorem is an injective immersion.*

Proof. We just saw that F is an immersion. The injectivity is similar but easier: the equation $f^m \circ F = F \circ g^m$ implies that if $F(\boldsymbol{x}_1) = F(\boldsymbol{x}_2)$, then

$$f^m \circ F \circ g^{-m}(\boldsymbol{x}_1) = f^m \circ F \circ g^{-m}(\boldsymbol{x}_2).$$

For m sufficiently large, $F \circ g^{-m}(\boldsymbol{x}_1) \neq F \circ g^{-m}(\boldsymbol{x}_2)$ when $\boldsymbol{x}_1 \neq \boldsymbol{x}_2$. Then use the assumption that f is injective. □

Note that we have now proved Theorem 1.

11. The Case of Stable Manifolds

If $f\colon X \to X$ is an automorphism, of course very stable manifolds behave just like very unstable manifolds, simply by considering f^{-1}. For the sake of completeness, we state this as Theorem 26. Note that for all of its being obvious, we are proving that in the case of attractive fixed points, the basins are Fatou-Bieberbach domains.

Theorem 26. *Let X be a complex manifold, $f\colon X \to X$ an automorphism, and $\boldsymbol{p} \in X$ a fixed point such that the hypotheses of Corollary 8 are satisfied. Then there is an injective immersion $F\colon \mathbb{C}^k \to X$ which parametrizes the k-very stable manifold of $\boldsymbol{p}$.*

But in the case where f is not an automorphism, stable and unstable manifolds are quite different, and in particular stable and very stable manifolds are not in general isomorphic to $\mathbb{C}^k$.

Example 27. Consider the polynomial $f(z) = \lambda z + z^2$ with $0 < |\lambda| < 1$. Then the stable manifold of 0 is the basin U of 0, the set of points $z \in \mathbb{C}$ such that $\lim_{m\to\infty} f^m(z) = 0$. This set is clearly bounded, and cannot be the image of any non-constant map $\mathbb{C} \to U$.

The appropriate statement about global very stable manifolds in general is Theorem 28.

Theorem 28. *Let X be a complex manifold, and $f\colon X \to X$ be an analytic mapping. Let $\boldsymbol{p} \in X$ be a fixed point satisfying the hypotheses of Corollary 8. Let E^L be the direct sum of the eigenspaces for the generalized eigenvalues $\lambda_1, \ldots, \lambda_k$, and suppose that $W \subset E^L$ and $F\colon W \to X$ are as in Corollary 8. Let $M = F(W)$ and*

$$\widetilde{M} = \bigcup_{m=0}^{\infty} f^{-m}(M).$$

Then there exists a resonant form $g\colon E^L \to E^L$ such that

$$\Psi = \lim_{m\to\infty} g^{-m} \circ F^{-1} \circ f^m \tag{9}$$

converges uniformly on compact subsets of $\widetilde{M}$ to a surjective analytic map $\widetilde{M} \to \mathbb{C}^k$ satisfying $\Psi \circ f = g \circ \Psi$.

Proof. Construct first a local coordinate near $\boldsymbol{p}$ in which the expression of f^{-1} (which exists near $\boldsymbol{p}$ by the inverse function theorem) is given in E^L by a resonant form g^{-1}. This can be done by Proposition 18. Now using these local coordinates, apply Corollary 8, to construct W and $F\colon W \to X$ locally parametrizing the k-very stable manifold of $\boldsymbol{p}$.

Now consider the limit of Equation (9). For any compact subset $K \subset \widetilde{M}$, there exists a number q such that $f^q(K) \subset M$. For all $m > q$, we then have

$$g^{-m} \circ F^{-1} \circ f^m = g^{-(q+1)} \circ F^{-1} \circ f^{(q+1)}.$$

Therefore the limit of Equation (9) does exist, uniformly on compact subsets. $\square$

Note that the mapping Ψ of Theorem 28 is usually not a ramified covering map, but more like a generic analytic mapping, with extremely complicated behavior near the boundary of $\widetilde{M}$.

We will spare the reader the applications of Corollary 12, showing that so long as the fixed point $\boldsymbol{p}$ does not bifurcate, the map Ψ can be made to depend analytically on parameters.

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