The Bott-Duffin synthesis

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The Bott-Duffin synthesis answers a basic question in electrical engineering:

How to make an electrical circuit that does what you want?

Bott on electrical circuits and differential geometry

Bott: No, because the actual work is just the same. When I worked with Duffin, it was mathematical thought; only the concepts were different. But the actual finding of something new seems to me the same. And you see, the algebraic aspects of network theory were an ideal introduction to differential geometry and the de Rham theory and to what Hermann Weyl was studying at the time, that is, harmonic theory. In effect, networks are a discrete version of harmonic theory. So when I came

A passive 1-port is a box containing

resistors, capacitors and inductors

(passive circuit elements),

with two wires sticking out (a port).

Such a circuit has an *impedance* $Z(s)$, defined as follows:

If you impose a current (using a current generator)

$$I(t) = e^{st}$$

through the port, the response voltage is

$$V(t) = Z(s)e^{st}.$$

Equivalently, if you impose the voltage $V(t) = e^{st}$ the resulting current is
$$I(t) = Y(s)e^{st}.$$

The function $Y(s) = \frac{1}{Z(s)}$ is called the *admittance* of the 1-port.

According to Bott, the first step to becoming an electrical engineeris to write
$$\sqrt{-1} = j.$$

Equivalently, if you impose the voltage $V(t) = e^{st}$ the resulting current is
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The physically meaningful values of s are the purely imaginary values $s = j\omega$, where ω is a frequency.

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Ending the synthesis

Suppose that $Z(j\omega) = Lj\omega$ for some $L > 0$.

Set $\quad Z_1(s) = \dfrac{Z(s)}{L}, \qquad$ and $\quad R(s) = \dfrac{sZ_1(s) - s_0^2}{s - Z_1(s)}.$

A bit of computation shows that

$$Z(s) = \frac{1}{\frac{1}{LR(s)} + \frac{1}{Ls}} + \frac{1}{\frac{R(s)}{Ls_0^2} + \frac{s}{Ls_0^2}}.$$

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The Bott-Duffin synthesis is the converse:

Every PRF is the impedance of a passive 1-port.

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3) Applies the Bott-Duffin synthesis to make the circuit.

I will assume my audience has forgotten any circuit theory they ever knew and start from scratch

Even from scratch, it is possible to give a complete proof in half an hour: the original paper is only one page.

Some basics

An electrical circuit is a graph, with each edge oriented, and each edge carrying a circuit element:

	Symbol	Equation	Units
• A resistor		$v = R\,i$	Ohms
• a capacitor		$C\,v' = i$	Farads
• an inductor		$L\,i' = v$	Henrys

To each edge e is associated a current i_e and a voltage drop v_e (the sign of both depends on the orientation of e).

These are subject to Kirchhoff's current law:

The sum of the currents at a node is zero

and to Kirchhoff's voltage law:

The sum of the voltage drops around any loop is zero

Let E be the set of edges, and let $\mathcal{I} \subset \mathbb{R}^E$ be the space of currents allowed by the Kirchhoff current law, and $\mathcal{V} \subset \mathbb{R}^E$ be the space of voltage drops allowed by the Kichhoff voltage law.

Tellegen's theorem. The spaces $\mathcal{I}$ and $\mathcal{V}$ are orthogonal complements in $\mathbb{R}^E$.

Proof. They are respectively the kernel of A and the image of $A^\top$, where A is the incidence matrix:

$$a_{i,j} = \begin{cases} 1 & \text{if node } i \text{ is the end of edge } j \\ -1 & \text{if node } i \text{ is the origin of edge } j \\ 0 & \text{otherwise.} \end{cases}$$

Using Kirchhoff's laws and Ohm's law for each resistor, we can eliminate many of the v_e and i_e

Under quite general circumstances,

–no loops of capacitors

–the inductors do not disconnect the circuit

the remaining variables from which all others can be deduced algebraically are the

currents in the inductors (and current generators), and the

voltages through the capacitors (and voltage generators)

A circuit of this sort is called normal

Suppose that a 1-port is normal, and

form a vector $\mathbf{x}(t)$ whose entries are these variables.

The time evolution of the circuit is described by an inhomogeneous linear differential equation of the form

$$\mathbf{x}' = A\mathbf{x} + \mathbf{f}(\mathbf{t})$$

where A is a square matrix, and $\mathbf{f}(t)$ is the imposed current, expressed in the variables of $\mathbf{x}$.

The eigenvalues of the matrix A are nonnegative

Differentiate the positive definite Hermitian form

inductors capacitors

$$E(\mathbf{i}, \mathbf{v}) = \frac{1}{2}\sum_{\lambda} L_\lambda |i_\lambda|^2 + \frac{1}{2}\sum_{\gamma} C_\gamma |v_\gamma|^2.$$

kinetic energy potential energy

The letters λ, γ and ρ are traditional for inductors, capacitors and resistors respectively.

We find

$$
\begin{aligned}
&\frac{d}{dt} E\big(\mathbf{i}(t), \mathbf{v}(t)\big) \\
&= \frac{1}{2}\frac{d}{dt}\left(\sum_\lambda L_\lambda |i_\lambda(t)|^2 + \sum_\gamma C_\gamma |v_\gamma(t)|^2\right) \\
&= \sum_\lambda L_\lambda \mathrm{Re}(\bar{i}_\lambda i'_\lambda) + \sum_\gamma C_\gamma \mathrm{Re}(\overline{v}_\gamma v'_\gamma) \\
&= \sum_\lambda \mathrm{Re}(\bar{i}_\lambda v_\lambda) + \sum_\gamma C_\gamma \mathrm{Re}(\overline{v}_\gamma i_\gamma) \\
&= -\sum_\rho \mathrm{Re}(\bar{i}_\rho v_\rho) = -\sum_\rho R_\rho |i_\rho|^2 \le 0.
\end{aligned}
$$

This certainly proves that the eigenvalues of A are nonpositive, and further that the purely imaginary eigenvalues are simple.

it also implies that if a 1-port is driven by a current $i(t) = e^{st}$ with Re $s>0$,

then the system of differential equations

$$\mathbf{x}' = A\mathbf{x} + \mathbf{f}(\mathbf{t})$$

can be solved by undetermined coefficients, setting

$$x_i(t) = a_i e^{st}$$

and solving for the a_i after cancelling a common factor e^{st}.

Proving the first half of the fundamental theorem of circuit theory

Z(s) is a PRF

Tellegen's theorem gives

$$-\bar{i}_G v_G = \sum_\lambda \bar{i}_\lambda v_\lambda + \sum_\gamma \bar{i}_\gamma v_\gamma + \sum_\rho \bar{i}_\rho v_\rho.$$

generator inductors capacitors resistors

Using undetermined coefficients, we can find the solution of the differential equation describing the circuit with $i_G = e^{st}$ as

$$v_G = -Z(s)e^{st}$$

$$i_\lambda = I_\lambda e^{st} \quad \text{so that} \quad v_\lambda = L_\lambda I_\lambda s e^{st}$$

$$v_\gamma = V_\gamma e^{st} \quad \text{so that} \quad i_\gamma = C_\gamma V_\gamma s e^{st}$$

$$i_\rho = I_\rho e^{st} \quad \text{so that} \quad v_\rho = R_\rho I_\rho \ \ e^{st}$$

Insert into

$$-\bar{i}_G v_G = \sum_\lambda \bar{i}_\lambda v_\lambda + \sum_\gamma \bar{i}_\gamma v_\gamma + \sum_\rho \bar{i}_\rho v_\rho$$

Cancel the common factor $|e^{st}|^2$ to get

$$Z(s) = s\sum_\lambda L_\lambda |I_\lambda|^2 + \bar{s}\sum_\gamma C_\gamma |V_\gamma|^2 + \sum_\rho R_\rho |I_\rho|^2.$$

Take real parts:

$$\mathrm{Re}Z(s) = \mathrm{Re}s\left(\sum_\lambda L_\lambda |I_\lambda|^2 + \sum_\gamma C_\gamma |V_\gamma|^2\right) + \sum_\rho R_\rho |I_\rho|^2.$$

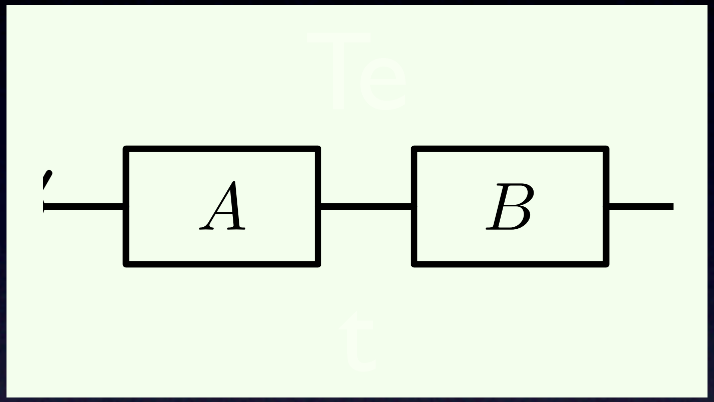
Everything on the right is ≥ 0 when $\mathrm{Re}s \geq 0$.

Proving the second half of the fundamental theorem of circuit theory

Every PRF is the impedance of some 1-port

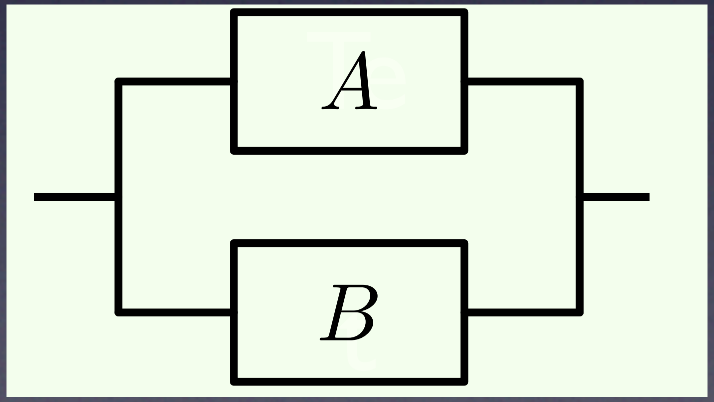
Bott and Duffin give a straightforward construction of an appropriate circuit.

Series and Parallel



If two 1-ports are connected in series their impedances add

$$Z(s) = Z_A(s) + Z_B(s)$$



If two 1-ports are connected in parallel their admittances add

$$Y(s) = Y_A(s) + Y_B(s)$$

$$\frac{1}{Z(s)} = \frac{1}{Z_A(s)} + \frac{1}{Z_B(s)}$$

Conditions on PRF's

(1) a) $\inf_{\mathrm{Re}s>0} \mathrm{Re}Z(s) = R > 0$

b) $\inf_{\mathrm{Re}s>0} \mathrm{Re}Y(s) = 1/R > 0$

(2) a) The function $Z(s)$ has a pole on the imaginary axis or at ∞

b) The function $Z(s)$ has a pole on the imaginary axis or at ∞

(3) a) There exists $\omega > 0$ and $L > 0$ with $Z(j\omega) = Lj\omega$

b) There exists $\omega > 0$ and $C > 0$ with $Y(j\omega) = Cj\omega$

Reduction in case 1

If a PRF satisfies either of conditions 1a or 1b

we can write $Z(s) = R + Z_1(s)$ or

$$Y(s) = \frac{1}{R} + Y_1(s)$$

and if we can find a circuit with impedance Z_1 or admittance Y_1

then putting it either in series (or in parallel) with a resistance R will realize Z (or Y).

The PRF Z_1 (or Y_1) belongs to case 2 or 3.

Reduction in case 2

This case uses partial fractions:
Since Z and Y are PRF's, any poles must be simple, with positive residues.

If W is a PRF, we can write $W(s) = P_W(s) + W_1(s)$

with $$P_W(s) = \frac{k_0}{s} + \sum_{i=0}^{m} \frac{2k_i s}{s^2 + \omega_i^2} + k_\infty s$$

with $k_0 \geq 0$, all $k_i \geq 0$ and $k_\infty \geq 0$

and W_1 is a PRF which is bounded on the right half-plane.

Thus if we can find circuits that realize the summands of P_W, and if we can realize W_1, we can realize W (as an impedance or as an admittance).

– The PRF $\frac{k_0}{s}$ is the impedance of a capacitor of capacitance $\frac{1}{k_0}$.

– The PRF $k_\infty s$ is the impedance of an inductor of inductance k_∞.

– The PRF $\frac{2ks}{s^2+\omega^2}$ is the impedance of a capacitor or capacitance $C = \frac{1}{2k}$ in parallel with an inductor of inductance $L = \frac{2k}{\omega^2}$.

Case 3 This case requires Richard's theorem

Richard's theorem. Let $W(s)$ be a PRF with no zeros or poles on the imaginary axis or at infinity.

a) There then exists a unique $s_0 \in \mathbb{R}_+$ with

$$W(s_0) = s_0.$$

b) The function

$$W_1(s) = \frac{sW(s) - s_0^2}{s - W(s)}$$

is a PRF with $\mathrm{rank} W_1(s) \leq \mathrm{rank} W(s)$.

c) If $W(j\omega) = j\omega$, then W_1 has a pole at $j\omega$.

Richard's theorem follows from Schwarz's lemma

a) The existence of s_0 follows from the intermediate value theorem. The uniqueness follows from Schwarz's lemma: a map from the right halfplane to itself that has more than one fixed point is the identity.

b) The map $\phi(s) = \dfrac{s - s_0}{s + s_0}$

maps the right half-plane to the unit disc, and $\phi(s_0) = 0$.

So $f = \phi \circ W \circ \phi^{-1}$

maps the unit disk to itself with $f(0) = 0$.

By Schwarz's lemma, so does $g(z) = f(z)/z$.

Thus $W_1 = \phi^{-1} \circ g \circ \phi$ is a PRF.

If you work it out, you will find

$$W_1(s) = \frac{sW(s) - s_0^2}{s - W(s)}.$$

c) Clearly if $W(j\omega) = j\omega$, then

$$W_1(s) = \frac{sW(s) - s_0^2}{s - W(s)} \quad \text{has a pole at } j\omega.$$

This completes the proof of Richard's theorem.

Ending the synthesis

Suppose that $Z(j\omega) = Lj\omega$ for some $L > 0$.

Set $$Z_1(s) = \frac{Z(s)}{L}, \quad \text{and} \quad R(s) = \frac{sZ_1(s) - s_0^2}{s - Z_1(s)}.$$

A bit of computation shows that

$$Z(s) = \frac{1}{\frac{1}{LR(s)} + \frac{1}{Ls}} + \frac{1}{\frac{R(s)}{Ls_0^2} + \frac{s}{Ls_0^2}}.$$

Suppose that the circuits C_1, C_2, C_3, C_4 have admittances

$$\frac{1}{LR(s)}, \ \frac{1}{Ls}, \ \frac{R(s)}{Ls_0^2}, \ \frac{s}{Ls_0^2}.$$

Then the circuit below has impedance $Z(s)$.