THE MONODROMY OF PROJECTIVE STRUCTURES

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Introduction

In this paper we shall give a new proof of the result, due to Hejhal [5], that the map associating to an isomorphism class of projective structures its conjugacy class of monodromy homomorphisms is a local homeomorphism.

We shall follow the following plan: show that the domain (Prop. 1) and the range (Prop. 4) are manifolds, identify their tangent spaces (Prop. 2 and 4), and compute the derivative of the map above. It turns out that one space is an Eichler cohomology space and the other is the cohomology of a group; they are canonically isomorphic by a classical theorem of algebraic topology. The derivative is the canonical isomorphism.

The idea of using differential calculus on this problem is not new: both Earle [2] and Gunning [4] have proposed similar proofs; this paper explains the appearance of Eichler cohomology in their computations. Many of the other results I establish in this paper were already known to Hejhal, Kra, Gunning, Maskit, Earle, Weil and no doubt others. The exposition is, I hope, in the spirit of Gunning's book and in fact the paper is largely a matter of putting parameters in arguments appearing there.

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NOTATION.

P^1 is the complex projective line (the Riemann Sphere)

$G = PGL_2(\mathbb{C}) = \mathrm{Aut}\, P^1$; $A \subset G$ is the subgroup of affine maps $z \mapsto az+b$.

$\mathcal{G} = pgl_2(\mathbb{C})$ = Space of analytic vector fields on P^1.

The adjoint action of G on $\mathcal{G}$ corresponds to the direct image of the corresponding vector fields.

We will speak of the fundamental group of a space only after a universal covering space has been chosen; the fundamental group is then the group of automorphisms of the universal covering space. All universal covering maps will be denoted u.

1. *Projective structures*

A *projective atlas* on a Riemann surface X is an open cover U_i of X and analytic maps $\alpha_i : U_i \to P_1$ which are homeomorphisms onto their images such that $\alpha_j \circ \alpha_i^{-1}$ is the restriction to $\alpha_i(U_i \cap U_j)$ of an element of G. Two projective atlases are equivalent if together they form a projective atlas; a *projective structure* on X is an equivalence class of projective atlases.

EXAMPLES. (i) If X is compact of genus ≥ 2 and H is the upper half plane, there is a covering map $u : H \to X$ by the uniformization theorem. Sections of u over simply-connected open subsets of X define a projective atlas.

(ii) Other planar covering spaces of X, such as the Schottky covering space, can be used to describe projective structures.

(iii) If $\Gamma \subset \mathbb{C}$ is a lattice and $X = \mathbb{C}/\Gamma$, appropriate restrictions of the canonical coordinate z of $\mathbb{C}$ define a projective structure on X as above; restrictions of e^{az} also do for any $a \in \mathbb{C}-\{0\}$. In these cases, the changes of coordinates are affine; a projective structure which can be defined by an affine atlas is called an *affine structure*.

Let α be a projective structure on a Riemann surface X, and $\tilde{X}$ be a universal covering space of X, with $u : \tilde{X} \to X$ the covering map.

LEMMA 1. (i) *There exists an analytic map* $f : \tilde{X} \to P^1$ *such that on any contractible open subset* $U \subset \tilde{X}$ *the composition* $f \circ u^{-1}$ *is a projective chart. Any other such map is of the form* $\sigma \circ f$ *for some* $\sigma \in G$.

(ii) *To every such* f *there corresponds a unique homomorphism* $\rho_f : \pi_1(X) \to G$ *such that* $\rho_f(\gamma) \circ f = f \circ \gamma$, *and* $\rho_{\sigma \circ f} = \sigma \circ \rho_f \circ \sigma^{-1}$.

Proof. Cover $\tilde{X}$ by open subset U_i on which u is injective, and such that there exist projective charts $\alpha_i : u(U_i) \to P^1$; let $\beta_i = \alpha_i \circ u^{-1}$. Then $\sigma_{ij} = \alpha_j \circ \alpha_i^{-1}$ is a 1-cocycle on $\tilde{X}$ with values in G. Since $\tilde{X}$ is contractible, this cocycle is a coboundary, after refining the cover if necessary, and there exist $\sigma_i \in G$ such that $\sigma_{ij} = \sigma_j^{-1} \circ \sigma_i$. Then on $U_i \cap U_j$, $\sigma_i \circ \alpha_i = \sigma_j \circ \alpha_j$ so all the $\sigma_i \circ \alpha_i$ are restrictions of a global map $f : \tilde{X} \to P^1$ with the appropriate properties. The second part of (i) is obvious.

In any U_i there is a homomorphism $\rho_{f,i} : \pi_1(X) \to G$ such that $\rho_{f,i}(\gamma) \circ f(x) = f(\gamma(x))$ for $x \in U_i$, since both $f \circ u^{-1}$ and $f \circ \gamma \circ u^{-1}$ are projective coordinates on X. But it is clear from analytic continuation that $\rho_{f,i}(\gamma) \circ f = f \circ \gamma$ on all of $\tilde{X}$, since $\tilde{X}$ is connected and both sides are analytic functions of x. Q.E.D.

Such a map f is called a developing map of X; ρ_f is the corresponding monodromy homomorphism.

We will need the following fact:

LEMMA 2. *A projective structure on a compact Riemann surface is equivalent to an affine structure if and only if the surface is of genus* 1.

Proof. See [3], p. 173. The result is purely topological, essentially saying that if a surface admits an affine structure, the cotangent bundle is trivial. Q.E.D.

COROLLARY. *If* X *is a compact Riemann surface of genus* ≥ 2 *and* α *is a projective structure on* X, *then the monodromy homomorphism* ρ_f *for any developing map* f *has non-commutative image.*

Proof. Any commutative subgroup of G is conjugate by an appropriate $\sigma \in G$ to a subgroup of the affine group $A \subset G$. The projective atlas formed by maps of the form $\sigma \circ f \circ u^{-1}$ on contractible open subsets of X is affine. Q.E.D.

The real interest of projective structures is the geometry of developing maps and the monodromy homomorphism. Beyond this corollary there is little to be said in general; the developing maps may fail to be covering spaces of their images, the monodromy homomorphisms may fail to be isomorphisms, and their images may fail to be discrete. In fact all of these pathologies occur for the family given in example (iii) for appropriate values of a.

2. *The Schwarzian derivative and the affine structure of* P(X)

Let U be a Riemann surface, $x \in U$, and f, g meromorphic functions on U such that $f'(x) \neq 0$, $g'(x) \neq 0$. There exists a unique $\sigma \in G$ such that f and $\sigma \circ g$ agree to order 2 at x. Then $d^3(f - \sigma \circ g)(x)$ is naturally a cubic map $T_x U \to T_{f(x)} P^1$, and $f'(x)^{-1} \circ d^3(f - \sigma \circ g)(x)$ is a cubic map $T_x U \to T_x U$. But for any one dimensional complex vector space V, the cubic maps $V \to V$ correspond naturally to the quadratic maps $V \to \mathbf{C}$. Therefore the construction above defines a quadratic form $S(f, g)(x)$ on $T_x U$, and it is easy to see that $S(f, g)$ is a meromorphic quadratic differential form on U, holomorphic at those points x where $f'(x) \neq 0$ and $g'(x) \neq 0$.

If $U \subset \mathbf{C}$ and z is the canonical coordinate on $\mathbf{C}$, then $S(f, z) = f''' f' - \frac{3}{2}(f'')^2/(f')^2$, as the classical definition requires.

For any Riemann surface U, let $Q(U)$ be the space of holomorphic quadratic forms on U.

LEMMA 3. *The Schwarzian derivative has the following properties*:

(i) $S(f, g) = S(f, \sigma \circ g) = S(\sigma \circ f, g)$ *for all* $\sigma \in G$.

(ii) $S(f, g) = 0$ *if* $f = \sigma \circ g$, *and conversely* $f = \sigma \circ g$ *if* $S(f, g) = 0$ *and* U *is connected.*

(iii) $S(f, g) + S(g, f) = S(f, h)$, *and* $S(f, g) = -S(g, f)$.

(iv) *If* U *is simply connected,* f *is schlicht on* U *and* $q \in Q(U)$, *there exists a solution* g *schlicht on* U *to the equation* $S(f, g) = q$.

Proof. Parts i, ii, iii are obvious. Part (iv) is similar to Lemma 1. Q.E.D.

Suppose the projective structures α and β on X are defined by atlases (U_i, α_i) and (V_j, β_j). Then the quadratic forms $S(\alpha_i, \beta_j)$, defined in $U_i \cap V_j$ coincide on open sets of form $U_{i_1} \cap U_{i_2} \cap V$ so they are induced by a quadratic form $q = \alpha - \beta$ on X.

Conversely, if $\alpha \in P(X)$ and $q \in Q(X)$, there is a unique projective structure $\beta \in P(X)$ such that $\beta - \alpha = q$, we shall denote it $q + \alpha$. If (U_i, α_i) is an atlas defining α and the U_i are simply connected, then β may be defined by (U_i, β_i) where the β are solutions of $S(\beta_i, \alpha_i) = q$ in U_i, which exist by Lemma 3, (iv).

LEMMA 4. *The map* $Q(X) \times P(X) \to P(X)$ *given by* $(q, \alpha) \mapsto q + \alpha$ *makes* $P(X)$ *into an affine space under* $Q(X)$.

Proof. All that is left to show is that $P(X)$ is not empty. This follows from the uniformization theorem, as in the example (i) §1, or from Riemann-Roch as in [3], p. 172. Q.E.D.

COROLLARY. *The space* $P(X)$ *is canonically a complex manifold, and for all* $\alpha \in P(X)$, *we have* $T_\alpha P(X) = Q(X)$.

3. *Relative projective structures*

Let $\pi : X \to S$ be a smooth family of compact Riemann surfaces parametrized by a complex manifold S (i.e. a proper analytic submersion with fibers $X(s) = \pi^{-1}(s)$ of dimension 1). A relative projective atlas on X is a relative atlas (U_i, α_i) where the U_i form an open cover of X, and the $\alpha_i : U_i \to P^1$ are analytic maps where restrictions to fibers of π are isomorphisms onto their images, such that over each $s \in S$, the pair $(U_i(s), \alpha_i(s))$ is a projective atlas on $X(s)$.

As above, two relative projective atlases are equivalent if together they still form a projective atlas, and a relative projective structure on X is an equivalence class of relative projective atlases.

REMARK. To say that a family of projective structures $\alpha(s)$ is induced by a relative structure is to say that $\alpha(s)$ depends analytically on s.

EXAMPLES. The family of projective structures obtained by applying the uniformization theorem fiber by fiber *does not* define a relative projective structure: the normalization requiring the images of the universal covering spaces to be the upper half plane cannot be made analytic.

The generalization of the uniformization theorem given by Bers [1] does give relative projective structures on the universal curve over Teichmüller space.

The canonical family of projective structure on $P(X) \times X \to P(X)$ is induced by a relative projective structure.

Let $P_S(X)$ be the set of pairs (s, α) such that $s \in S$ and α is a projective structure on $X(s)$.

PROPOSITION 1. (i) *There is a unique structure of a complex manifold on $P_S(X)$ such that the projection $\rho : P_S(X) \to S$ given by $(s, \alpha) \mapsto s$ is analytic, and analytic families of projective structures on X given by section of* p *are induced by relative projective structures on* X.

(ii) *The action of $Q_S(X)$ on $P_S(X)$ over S given by $((s, q), (s, \alpha)) \mapsto (s, q+\alpha)$ makes $P_S(X)$ into an analytic affine bundle over* S, *under the analytic vector bundle $Q_S(X)$.*

Proof. The proposition is clearly local in S. Suppose a relative projective structure α on X can be found (even locally over small subset of S).

Then the map $(s, q) \to (s, q + \alpha(s))$ is a bijection $Q_S(X) \to P_S(X)$. Give $P_S(X)$ the induced structure; with this structure, $P_S(X)$ clearly satisfies (ii).

To see that it satisfies (i), we need to know that if q is a section of $Q_S(X)$, then the family of projective structures induced by $\alpha(s) + q(s)$ is induced by a relative projective structure.

Let (U_i, α_i) be a relative projective atlas defining α. On $U_i(s)$, the equation $S(\alpha_i(s), \beta_i) = q(s)$ is an analytic differential equation of third order depending analytically on s, whose solutions will exist in $U_i(s)$ by Lemma 3, (iv) if the U_i are sufficiently small, and will depend analytically on s if initial conditions are chosen analytic in s. But this can be done, for instance by picking (locally in S) a section $S \to X$ of π and requiring $\beta_i(s)$ to coincide with $\alpha_i(s)$ to order 2 along the section. Clearly the β_i form a relative projective atlas with the desired properties.

Thus we are left with showing that over sufficiently small open subsets of S, X carries a relative projective structure. This may be shown by appealing to the universal property of Teichmüller space and the simultaneous uniformization theorem of Bers [1]; we shall prove a slightly more general result.

LEMMA 5. *Let $\pi : X \to S$ be a proper and smooth family of Riemann surfaces of genus at least 2, with S a Stein manifold. Then X admits relative projective structures.*

Proof. Let (U_i, ϕ_i) be any relative atlas (relative atlases exist by the implicit function theorem). On the $U_i \cap U_j$, consider the section of $\Omega^{\otimes 2}_{X/S}$ given by $S(\phi_i(s), \phi_j(s))$. These define a 1-cochain with values in $\Omega^{\otimes 2}_{X/S}$ for the cover $\{U_i\}$ which is a cocycle by Lemma 2, iii.

But the Leray spectral sequence of π gives an exact sequence

$$0 \to H^0(S, R^1\pi_*\Omega^{\otimes 2}_{X/S}) \to H^1(X, \Omega^{\otimes 2}_{X/S}) \to H^1(S, \pi_*\Omega^{\otimes 2}_{X/S})$$

and the first term is zero because $H^1(X(s), \Omega^{\otimes 2}) = 0$ by Riemann-Roch; the last term is zero because S is Stein.

Therefore refining the cover if necessary, we may assume that there are sections q_i of $\Omega^{\otimes 2}_{X/S}$ over U_i such that

$$q_i - q_j = S(\phi_i, \phi_j)\,.$$

Solutions α_i of the differential equations $S(\alpha_i, \phi_i) = q_i$ chosen so as to satisfy some analytic initial condition such as to agree with ϕ_i to order 2 along some section $S \to U_i$ of π, will then form a projective atlas for X, perhaps after further refining the cover to make them injective on fibers. Q.E.D.

COROLLARY. (i) *The family of projective structures on the family of Riemann surfaces* $p^*X \to P_SX$ *which is* α *on the fiber over* $\alpha \in P_SX$ *is induced by a canonical relative projective structure* $\alpha_S(X)$.

(ii) *The space* P_SX *has the following universal property: The map which associates to any analytic mapping* $f: T \to P_S(X)$ *the projective structure* $f^*\alpha_S(X)$ *on the family of Riemann surfaces* $(p \circ f)^*X$ *is a bijection of* $\mathrm{Mor}(T, P_S(X))$ *onto the set of relative projective structures on* $(p \circ f)^*X$.

The proof is left to the reader.

4. *Infinitesimal deformations and Eichler cohomology*

In this paragraph, we shall carry out an infinitesimal deformation theory for projective structures analogous to the Kodaira-Spencer theory for complex structures.

Let U be a Riemann surface, α be a projective structure on U and χ an analytic vector field on U. Choose a one-parameter family of maps $\phi_t: U \to U$ with $\phi_0 = \mathrm{id}$ and $\phi_0' = \chi$, and define the Lie derivative of α in the direction χ

$$L_\chi(\alpha) = \lim_{t \to 0} \frac{\phi_t^*(\alpha) - \alpha}{t}\,.$$

Clearly $L_\chi(\alpha)$ is an analytic quadratic form on U, and we leave it to the reader to prove that if ζ is a projective coordinate on U and $\chi = \chi(\zeta)\frac{\partial}{\partial\zeta}$, then $L_\chi(\alpha) = \chi'''(\zeta)d\zeta^2$. In particular, the Lie derivative does not depend on the family ϕ_t that was chosen.

Let Λ_α be the subsheaf of the sheaf of germs of analytic vector fields which is the kernel of the morphism $\psi \mapsto L_\chi(\alpha)$; we then obtain an exact sequence of sheaves

$$0 \longrightarrow \Lambda_\alpha \longrightarrow TU \xrightarrow{L(\alpha)} \Omega^{\otimes 2} \longrightarrow 0$$

which will be important; TU stands for the sheaf of germs of vector fields on U (as opposed to the tangent bundle TU), and the Lie derivative is surjective because of the formula that computes it in a projective coordinate.

REMARKS. In a projective coordinate ζ, sections of Λ_α are exactly those vector fields which can be written $p(\zeta)\frac{\partial}{\partial\zeta}$ with p a polynomial of degree at most two; such vector fields are called infinitesimal automorphisms of α because the flows they generate send α to itself.

The sheaf Λ_α is locally constant of rank 3, an example of what topologists call a local system. In particular, it may be thought of as the sheaf of germs of sections of a covering space, with fiber isomorphic to $\mathbf{C}^3$ with the discrete topology.

Let $X \to S$ a family of Riemann surfaces, $s_0 \in S$ and X_0 the surface above it. Suppose α is a relative projective structure on X which induces the projective structure α_0 on X_0. We shall describe a linear map $T_{s_0}S \to H^1(X_0, \Lambda_{\alpha_0})$ which measures the infinitesimal deformation of α.

Let (U_i, α_i) be a relative projective atlas on X; by restricting S and refining the cover $\mathfrak{U} = \{U_i\}$ we may assume that for each i and all $s \in S$ the maps $\alpha_i(s): U_i(s) \to \mathbf{P}^1$ are homeomorphisms onto some open set $V_i \subset \mathbf{P}^1$. Define $\phi_i(s) = \alpha_i(s)^{-1} \circ \alpha_i(s_0)$ and $\phi_{i,j}(s) = \phi_i(s)^{-1} \circ \phi_j(s)$, where $\phi_{i,j}(s)$ is defined in an open subset of $U_i \cap U_j$ which will include any given point for s sufficiently near s_0. The maps $\phi_{i,j}(s)$ satisfy the following two identities:

$$\phi_{i,j}(s) \circ \phi_{j,k}(s) = \phi_{i,k}(s)$$

$$\phi_{i,j}(s)^* \alpha_0 - \alpha_0 = 0 ,$$

the first obviously and the second because (U_i, α_i) is a relative *projective* atlas.

Since $\phi_{i,j}(s_0)$ is the identity map $U_i \cap U_j \to U_i \cap U_j$, the derivative $d_{s_0}\phi_{i,j}(v) = \chi_{i,j}(v)$ is a vector field on $U_i \cap U_j$ for any $v \in T_{s_0}S$, and the derivatives of the identities above give:

$$\chi_{i,j}(v) + \chi_{j,k}(v) = \chi_{i,k}(v)$$

$$L_{\chi_{i,j}}(\alpha_0) = 0 .$$

The first identity says that $\chi(v) = \{\chi_{i,j}(v)\}$ is a cocycle in $C^1(\mathfrak{U}, TX_0)$, and the second that it is in fact in $C^1(\mathfrak{U}, \Lambda_{\alpha_0})$. Define the *infinitesimal deformation* of α at s_0

$$d_{s_0}\alpha : T_{s_0}S \to H^1(X_0, \Lambda_{\alpha_0})$$

by $d_{s_0}(\alpha)(v) =$ the cohomology class of $\chi(v)$. We leave it to the reader to prove that the class does not depend on the projective atlas that was chosen.

The map $d_{s_0}(\alpha)$ has the following properties:

(i) *It commutes with change of basis*, i.e., if $f : T \to S$ is a map with $f(t_0) = s_0$ and we give f^*X the relative projective structure $f^*\alpha$, then $d_{t_0}(f^*\alpha) = d_{s_0}(\alpha) \circ d_{t_0}f$.

(ii) *The map* $i_* \circ d_{s_0}\alpha : T_{s_0}S \to H^1(X_0, TX_0)$ *obtained by composing* $d_{s_0}\alpha$ *with the map* $H^1(X_0, \Lambda_{\alpha_0}) \to H^1(X_0, TX_0)$ *induced by the inclusion*

$\Lambda_{\alpha_0} \subset TX_0$ *is the Kodaira-Spencer map classifying the deformation of the complex structure of* X *at* x_0.

(iii) Let X_0, α_0 be any compact Riemann surface with a projective structure, and let α be the canonical relative projective structure on $P(X_0) \times X_0 \to P(X_0)$. *Then* $T_{\alpha_0}P(X_0) = Q(X_0)$, *and* $d_{\alpha_0}\alpha : Q(X_0) \to H^1(X_0, \Lambda_{\alpha_0})$ *is the "connecting homomorphism" coming from the long exact sequence associated to the short exact sequence* (1).

Part (i) follows immediately from the construction, (ii) is clear since the cocycle $\chi(v)$ is a definition of the Kodaira-Spencer map [7], (iii) is a computation we shall leave to the reader.

Now let us examine the universal case; let M be a compact surface of genus at least 2, Θ_M the Teichmüller space modelled on M and $\pi : \Xi_M \to \Theta_M$ the universal Teichmüller curve. We shall omit the subscript M in the sequel.

Let $p : P_\Theta \Xi \to \Theta$ be the canonical projection and give $p^*\Xi$ the canonical relative projective structure α given by the corollary to Proposition 1.

PROPOSITION 2. *Let* θ_0 *be a point in* Θ, $X_0 = \pi^{-1}(\theta_0)$ *the Riemann surface above it and* $\alpha_0 \in P(X_0)$. *Then*

$$d_{\alpha_0}\alpha : T_{\alpha_0}P_\Theta\Xi \to H^1(X_0, \Lambda_{\alpha_0})$$

is an isomorphism.

Proof. Consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & T_{\alpha_0}P(X_0) \arrow[r] \arrow[d, "\mathrm{id}"] & T_{\alpha_0}P_\Theta\Xi \arrow[r, "d_{\alpha_0}p"] \arrow[d, "d_{\alpha_0}\alpha"] & T_{\theta_0}\Theta \arrow[r] \arrow[d, "\wr"] & 0 \\
H^0(X_0, \Omega^{\otimes 2}) \arrow[r] & H^1(X_0, \Lambda_{\alpha_0}) \arrow[r, "i_*"] & H^1(X_0, TX_0) \arrow[r] & 0
\end{tikzcd}$$

The top line is induced by the inclusion of $P(X_0)$ as the fiber of p above θ_0 (cf. Prop. 1), the bottom line is extracted from the long exact sequence associated to the short exact sequence (1), the left vertical map is the isomorphism of the corollary to Lemma 4 and the right vertical map is the Kodaira-Spencer isomorphism. The left-hand square commutes by properties (i) and (iii) of $d_{\alpha_0}\alpha$ and the right-hand square by property (ii). The proposition now follows from the five lemma. Q.E.D.

REMARK. Proposition 2 identifies the tangent space to the space $P_\Theta\Xi$ of "all projective structures on all Riemann surfaces" exactly in the same sense that the Kodaira-Spencer isomorphism identifies the tangent space to the space Θ of "all Riemann surfaces" as $T_{\theta_0}\Theta = H^1(X_0, TX_0)$.

5. *The space* Hom(Γ, G)

Let Γ be the fundamental group of a surface, given by generators $\Sigma = \{a_1, \cdots, a_{2g}\}$ subject to the one relation

$$\prod_{i=1}^{g} [a_i, a_{i+g}] = 1 .$$

Clearly the set $\mathrm{Hom}(\Gamma, G)$ may be identified with the subset of G^{2g} defined by the analytic equation $f = 1$ where $f : G^{2g} \to G$ is given by $f(\sigma_1, \cdots, \sigma_{2g}) = \prod_{i=1}^{g} [\sigma_i, \sigma_{i+g}]$. This gives $\mathrm{Hom}(\Gamma, G)$ the structure of an analytic space.

LEMMA 6. *With this analytic structure,* $\mathrm{Hom}(\Gamma, G)$ *has the following universal property: for any analytic space* S, *morphisms* $S \to \mathrm{Hom}(\Gamma, G)$ *correspond bijectively to morphisms* $S \times \Gamma \to G$ *which are analytic, and whose restrictions to* $\{s\} \times \Gamma$ *are group homomorphisms for all* $s \in S$.

REMARK. In this lemma Γ is considered as a discrete analytic space. The proof is trivial and left to the reader. In particular, the analytic structure on $\mathrm{Hom}(\Gamma, G)$ does not depend on the chosen presentation. We may therefore expect that there is an intrinsic description of the local structure of $\mathrm{Hom}(\Gamma, G)$, in particular of its tangent space, etc. The object of this paragraph is to give such a description.

Let $\mathrm{Hom}^*(\Gamma, G) \subset \mathrm{Hom}(\Gamma, G)$ be the open set of representations with non-commutative image.

For any representation $\rho : \Gamma \to G$, we may consider $\mathfrak{g}$ as a Γ-module by $\gamma \cdot \xi = \mathrm{Ad}\, \rho(\gamma)(\xi)$; we shall denote this Γ-module $\mathfrak{g}_\rho$. Recall [6] that a derivation $\delta : \Gamma \to \mathfrak{g}_\rho$ is a map satisfying $\delta(\gamma_1\gamma_2) = \delta(\gamma_1) + \gamma_1 \cdot \delta(\gamma_2)$ and that those derivations of the form $\delta_\xi(\gamma) = \xi - \gamma \cdot \xi$ are called principal derivations. Call $\mathrm{Der}(\Gamma, \mathfrak{g}_\rho)$ the space of derivations and $\mathrm{IDer}(\Gamma, \mathfrak{g}_\rho)$ the subspace of principal derivations. A classical description of $H^1(\Gamma, \mathfrak{g}_\rho)$ is $\mathrm{Der}(\Gamma, \mathfrak{g}_\rho)/\mathrm{IDer}(\Gamma, \mathfrak{g}_\rho)$; this is the description we shall use.

The tangent space to G at any σ is $\mathfrak{g}$ (in two different ways); we shall use the local chart $\mathfrak{g} \to G$ of G near σ given by $\xi \mapsto \exp(\xi)\sigma$.

PROPOSITION 3. *The space* $\mathrm{Hom}^*(\Gamma, G)$ *is a submanifold of* G^{2g}.

For any $\rho \in \mathrm{Hom}^*(\Gamma, G)$ *the map* $\mathrm{Der}(\Gamma, \mathfrak{g}_\rho) \to \mathfrak{g}^{2g}$ *given by* $\delta \mapsto \delta|_\Sigma$ *is an isomorphism of* $\mathrm{Der}(\Gamma, \mathfrak{g}_\rho)$ *onto* $T_\rho \mathrm{Hom}(\Gamma, G)$.

Proof. A computation which begins

$$\Pi\, [e^{\xi_i}\sigma_i, e^{\xi_{i+g}}\sigma_{i+g}] = e^{\xi_1}(\sigma_1 e^{\xi_{1+g}}\sigma_1^{-1})(\sigma_1\sigma_{1+g}\sigma_1^{-1}e^{-\xi_1}\sigma_1\sigma_{1+g}^{-1}\sigma_1^{-1}) \cdots$$
$$= e^{\xi_1} e^{\mathrm{Ad}\sigma_1 \cdot \xi_{1+g}} e^{-\mathrm{Ad}(\sigma_1\sigma_{1+g}\sigma_1^{-1})\cdot \xi_1} \cdots$$

and ends using $e^{\chi_1}e^{\chi_2} = e^{\chi_1+\chi_2} + O(|\chi_1|^2 + |\chi_2|^2)$ shows that the derivative of f at $\boldsymbol{\sigma} = (\sigma_1, \cdots, \sigma_{2g}) \in G^{2g}$ in the direction $\boldsymbol{\xi} = (\xi_1, \cdots, \xi_{2g}) \in \mathfrak{g}^{2g}$ is

$$d_{\boldsymbol{\sigma}} f(\boldsymbol{\xi}) = \sum_{i=1}^{g} \prod_{j=1}^{i-1} [\sigma_j, \sigma_{j+g}] \cdot ((1 - \sigma_i\sigma_{i+g}\sigma_i^{-1}) \cdot \xi_i + (\sigma_i - [\sigma_i, \sigma_{i+g}]) \cdot \xi_{i+g}).$$

If $\sigma = \rho|_\Sigma$ for some $\rho \in \mathrm{Hom}(\Gamma, G)$, essentially the same computation shows that $d_\sigma f(\xi) = 0$ is the necessary and sufficient condition for $\xi : \Sigma \to \mathfrak{g}_\rho$ to extend (obviously uniquely) to a derivation $\Gamma \to \mathfrak{g}_\rho$.

Thus all we need to prove is that $d_\sigma f : \mathfrak{g}_\rho^{2g} \to \mathfrak{g}_\rho$ is surjective if $\rho \in \mathrm{Hom}^*(\Gamma, G)$. The basic fact (left to the reader) is that if $\tau_1, \tau_2 \in G$ do not commute, the linear map $\mathfrak{g}_\rho \times \mathfrak{g}_\rho \to \mathfrak{g}_\rho$ given by $(\xi_1, \xi_2) \mapsto (1-\tau_1)\xi_1 + (1-\tau_2)\xi_2$ is surjective.

This result gets applied twice. First suppose that for some i, $1 \le i \le g$, σ_i and σ_{i+g} do not commute. Then since

$$(1-\sigma_i\sigma_{i+g}\sigma_i^{-1})\xi_i + (\sigma_i - [\sigma_i, \sigma_{i+g}])\xi_{i+g}$$

$$= \sigma_i((1-\sigma_{i+g})\sigma_i^{-1} \cdot \xi_i + (1-\sigma_{i+g}\sigma_i^{-1}\sigma_{i+g}^{-1})\xi_{i+g})$$

the images of the ξ_i and ξ_{i+g} already fill out the image of $d_\sigma f$.

If each σ_i commutes with σ_{i+g} the expression for the derivative of f simplifies to

$$d_\sigma f(\xi) = \sum_{i=1}^{g} ((1-\sigma_{i+g}) \cdot \xi_i + (\sigma_i - 1) \cdot \xi_{i+g}))$$

and the result is clear since for some i, j, σ_i and σ_j do not commute.

Q.E.D.

REMARK. This result is a special case of the following more general results: If Γ is a group of finite presentation, G is a Lie group with Lie algebra $\mathfrak{g}$, then $\mathrm{Hom}(\Gamma, G)$ is an analytic space, its Zariski tangent space at ρ is $\mathrm{Der}(\Gamma, \mathfrak{g}_\rho)$ and the equations defining locally $\mathrm{Hom}(\Gamma, G)$ in $\mathrm{Der}(\Gamma, \mathfrak{g}_\rho)$ may be chosen to have values in $H^2(\Gamma, \mathfrak{g}_\rho)$. In our case, this boils down to the fact that if the image of ρ is not commutative, $H^2(\Gamma, \mathfrak{g}_\rho) = 0$, which may be proved by Poincaré duality.

PROPOSITION 4. (i) *The group* G *acts freely on* $\mathrm{Hom}^*(\Gamma, G)$, *and the quotient* $\mathrm{Hom}^*(\Gamma, G)/G$ *has a unique structure of an analytic manifold such that the projection* $\mathrm{Hom}^*(\Gamma, G) \to \mathrm{Hom}^*(\Gamma, G)/G$ *is analytic.*

(ii) *For any* $\rho \in \mathrm{Hom}^*(\Gamma, G)$, *the derivative of the inclusion* $G \to \mathrm{Hom}^*(\Gamma, G)$ *given by* $\sigma \mapsto \sigma \circ \rho \circ \sigma^{-1}$ *at* $1 \in G$ *is the map* $\mathfrak{g}_\rho \to \mathrm{Der}(\Gamma, \mathfrak{g}_\rho)$ *given by* $\xi \to \delta_\xi$. *In particular the tangent space to* $\mathrm{Hom}^*(\Gamma, G)/G$ *at the image of* ρ *is canonically isomorphic to* $H^1(\Gamma, \mathfrak{g}_\rho)$.

Proof. The fact that G acts freely follows from the fact that commuting is an equivalence relation on non-trivial elements of G. This may for instance be seen by observing that $\sigma_1 \neq 1$ and $\sigma_2 \neq 1$ commute if and only if they have the same fixed points in $\mathbf{P}^1$.

Similarly, if ρ_1 and ρ_2 are not conjugate, they have neighborhoods U_1 and U_2 such that no element of U_1 is conjugate to an element of U_2, since the fixed points of a non-trivial $\gamma \in G$ vary continuously with γ. Therefore the graph of the equivalence relation is closed, and the quotient is Hausdorff. The existence and uniqueness of the analytic structure follows from the analyticity of the action of G, and the derivative in (ii) is computed from $e^{\xi}\rho e^{-\xi} = e^{\xi-\rho\cdot\xi}\rho + 0(|\xi|^2)$. Q.E.D.

6. *Hejhal's theorem*

Let $\pi: X \to S$ be a family of Riemann surfaces with a relative projective structure α. Suppose S is contractible and let $\Gamma = \pi_1(X) = \pi_1(X(s))$. If π admits analytic sections, there are relative developing maps $f: \tilde{X} \to \mathbf{P}^1$, i.e., analytic maps which, restricted to $\tilde{X}(s)$, are developing maps of $\alpha(s)$. This is just a matter of picking an analytic normalization, for instance requiring that f should agree to order 2 with a relative analytic chart along a section.

REMARK. The requirement that π admit analytic sections is too stringent. In fact, there are relative developing maps if S is contractible and Stein. Indeed, the space of all developing maps of the $X(s)$ forms a principle analytic bundle under G over S, and so is trivial by Grauert's theorem if S is Stein and contractible. This applies in particular to the universal family over $P_\Theta \Xi$.

Clearly if $f: X \to P^1$ is a relative developing map, the associated $\rho_f: S \to \mathrm{Hom}(\Gamma, G)$ which associates to each $s \in S$ the monodromy homomorphism of $f(s)$, is analytic.

Let $F: P_\Theta \Xi \to \mathrm{Hom}(\Gamma, G)/G$ be induced by the above construction, for the universal family of projective structures parametrized by $P_\Theta \Xi$.

REMARK. The global existence of F does not require Grauert's theorem, because we have divided by the action of G. It does require the contractibility of $P_\Theta \Xi$, so that a universal covering space $\widetilde{p^*\Xi}$ induces a universal covering space $p^*\Xi$ over each fiber of $p^*\Xi \to P_\Theta \Xi$.

THEOREM. *The map* F *is an analytic local homeomorphism.*

The fact that F is analytic follows from the fact that F lifts locally (and even globally by Grauert) to an analytic map $P_\Theta \Xi \to \mathrm{Hom}(\Gamma, G)$.

By the corollary to Lemma 2, the image of a monodromy homomorphism is never commutative, and so both the range and the image of F are manifolds, whose tangent spaces we know.

Let $\alpha_0 \in P_\Theta \Xi$ be a projective structure on X_0; let $f: X_0 \to P^1$ be a developing map for α_0 and $\rho: \Gamma \to G$ its monodromy homomorphism. The theorem will now follow from

LEMMA 6. *The derivative* $d_{\alpha_0} F: H^1(X_0, \Lambda_{\alpha_0}) \to H^1(\Gamma, \mathfrak{g}_\rho)$ *is an isomorphism.*

The two tangent spaces look similar; they are in fact canonically isomorphic by the classical theorem of algebraic topology which says that one way to compute the cohomology of a group Γ with values in a Γ-module is to compute the cohomology of a $K(\Gamma, 1)$, with coefficients in the associated local system in the sense of the following lemma.

LEMMA 7. *The group* Γ *acts on* $\tilde{X}_0 \times \mathfrak{g}_\rho$ *by* $\gamma \cdot (x, \xi) = (\gamma \cdot x, \rho(\gamma)_* \xi)$, *and the map* $\tilde{X}_0 \times \mathfrak{g}_\rho \mapsto \Lambda_{\alpha_0}$ *given by* $(x, \xi) \mapsto (u(x), u_* f^* \xi)$ *induces an isomorphism on the quotient.*

The proof is immediate and left to the reader.

We cannot unfortunately use the canonical isomorphism without explicitly constructing it. There are many ways to do this; the one we shall use here is adapted to our knowledge of the two spaces, one via Čech cocycles and the other via derivations.

It is possible to compute Čech cohomology using a generalization of an open cover: an etale cover. The "open sets" are manifolds U_i and immersions $U_i \to X$, whose images are required to cover X. The intersections $U_i \cap U_j$ must be replaced by the fiber products $U_i \times_X U_j$, and similarly for multiple intersections.

Moreover, Leray's theorem still applies: if the U_i as well as all their fiber products are cohomologically trivial (for whatever sheaf we may be considering) the Čech cohomology for that cover is the cohomology of the sheaf (either in the Čech sense of direct limit over all covers, or via resolutions, or whatever, which are all isomorphic).

We shall apply this to the cover consisting of a single open set $\tilde{X} \to X$. The map $\tilde{X} \times_X \tilde{X} \times_X \tilde{X} \times \cdots \times_X \tilde{X} = X \times \Gamma^n$ given by $(x, \gamma_1, \cdots, \gamma_n) \mapsto (x, \gamma_1(x), \cdots, \gamma_n(x))$ and the identification of Lemma 7 give isomorphisms $C^n(X, \tilde{X}; \Lambda_\alpha) = \mathfrak{g}_\rho^{\Gamma^n}$. In fact the complex is the classical inhomogeneous bar complex [6] whose first two differentials are

$$d_0 : \mathfrak{g}_\rho \to \mathfrak{g}_\rho^\Gamma \text{ given by } d_0(\xi)(\gamma) = \xi - \gamma \cdot \xi ;$$

$$d_1 : \mathfrak{g}_\rho^\Gamma \to \mathfrak{g}_\rho^{\Gamma \times \Gamma} \text{ given by } d_1\chi(\gamma_1, \gamma_2) = \chi(\gamma_1) + \gamma_1\chi(\gamma_2) - \chi(\gamma_1\gamma_2).$$

In particular, the kernel of d_1 is formed of the derivations $\Gamma \to \mathfrak{g}_p$ and the image of d_0 is formed of the principal derivations.

In our case, X_0 is a $K(\Gamma, 1)$ so Leray's theorem applies to guarantee that the cohomology of the complex is in fact $H^1(X_0, \Lambda_{\alpha_0})$.

LEMMA 6′. *The derivative* $d_{\alpha_0}F$ *is the isomorphism* $H^1(X_0, \Lambda_{\alpha_0}) \to H^1(\Lambda, \mathfrak{g}_\rho)$ *described above.*

Proof. Choose an analytic curve $\alpha(t)$ in $P_\Theta \Xi$; let f_t be a relative developing map and ρ_t the corresponding monodromy homomorphisms. Define (as in the construction of §4) a family of analytic maps $\phi_t : U_t \to X(t)$ which:

a) are analytic isomorphisms onto their images, and analytic in t;

b) are defined in subsets $U_t \subset \tilde{X}_0$ which fill out $\tilde{X}_0$ as t becomes small;

c) satisfy $f_0 = f_t \circ \phi_t$ in U_t, and $\phi_0 =$ identity of $\tilde{X}_0$.

Then $\frac{d}{dt}\alpha(t)\Big|_{t=0}$ is represented by the Čech cocycle for the cover $\tilde{X}_0$ which is, on the component $\tilde{X}_0 \times \{\gamma\}$ of $\tilde{X}_0 \times_{X_0} \tilde{X}_0$, given by $\xi_\gamma = \frac{d}{dt}(\phi_t^{-1} \circ \gamma \circ \phi_t)\Big|_{t=0}$.

Using $f_t \circ \gamma = \rho_t(\gamma) f_t$ the expression above may be written $\xi_\gamma = \frac{d}{dt}(f_0^{-1} \circ \rho_t(\gamma) \circ f_0)\Big|_{t=0}$, where the entire expression $f_0^{-1} \circ \rho_t(\gamma) \circ f_0$ is defined in U_t.

If we write $\frac{d}{dt}\rho_t(\gamma)\Big|_{t=0} = \xi'_\gamma$ (it is best to think of ξ'_γ as a vector field on P^1), then differentiating the expression above gives $\xi_\gamma = f_0^* \xi'_\gamma$. This is the identification of Lemma 7. Q.E.D.

REMARKS. Some obvious questions, unsolved to the author's knowledge, are:

What is the image of F?

What do the fibers of F look like, and their projections in Teichmüller space? It is known [8] that F is not injective, but it is injective on fibers [3].

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