0.3 Set Theory

There is nothing new about the concept of a "set" composed of elements such that some property is true. Euclid spoke of geometric *loci*, a locus being the set of points defined by some property. But historically, mathematicians apparently did not think in terms of sets, and the introduction of set theory was part of a revolution at the end of the nineteenth century that included topology and measure theory; central to this revolution was Cantor's discovery (discussed in Section 0.6) that some infinities are bigger than others.

In the late 1960s and early 1970s, set theory was taught in secondary and even elementary schools as a subject in its own right: mountains were made of molehills. This was a resounding failure, and many schools have gone to the opposite extreme, some dropping the subject altogether.

The Latin word *locus* means "place"; its plural is *loci*.

In spoken mathematics, the symbols $\in$ and $\subset$ often become "in": $\mathbf{x} \in \mathbb{R}^n$ becomes "x in Rn," and $U \subset \mathbb{R}^n$ becomes "U in Rn." Make sure you know whether "in" means element or subset.

At the level at which we are working, set theory is a language, with a vocabulary consisting of seven words:

$\in$	"is an element of"
$\{a \mid p(a)\}$	"the set of a such that $p(a)$ is true"
$\subset$	"is a subset of": $A \subset B$ means that every element of A is an element of B. Note that with this definition, every set is a subset of itself: $A \subset A$.
$\cap$	"intersect": $A \cap B$ is the set of elements of both A and B.
$\cup$	"union": $A \cup B$ is the set of elements of either A or B or both.
$\times$	"cross": $A \times B$ is the set of pairs (a, b) with $a \in A$ and $b \in B$.
$-$	"complement": $A - B$ is the set of elements in A that are not in B.

You should think that *set, subset, intersection, union,* and *complement* mean precisely what they mean in English. However, this suggests that any property can be used to define a set. We will see, when we discuss Russell's paradox, that this is too naive. But for our purposes, naive set theory is sufficient.

One set has a standard name: the empty set ϕ, which has no elements. There are also sets of numbers that have standard names; they are written in *blackboard bold*, a font we use only for these sets:

$\mathbb{N}$ is for "natural," $\mathbb{Z}$ is for "Zahl," the German for number, $\mathbb{Q}$ is for "quotient," $\mathbb{R}$ is for "real," and $\mathbb{C}$ is for "complex."

When writing with chalk on a blackboard, it's hard to distinguish between normal letters and bold letters. Blackboard bold font is characterized by double lines, as in $\mathbb{N}$ and $\mathbb{R}$.

$\mathbb{N}$	"the natural numbers" $\{0, 1, 2, \ldots\}$
$\mathbb{Z}$	"the integers," i.e., signed whole numbers $\{\ldots, -1, 0, 1, \ldots\}$
$\mathbb{Q}$	"the rational numbers" p/q, with $p, q \in \mathbb{Z}$, $q \neq 0$
$\mathbb{R}$	"the real numbers," which we will think of as infinite decimals
$\mathbb{C}$	"the complex numbers" $\{a + ib \mid a, b \in \mathbb{R}\}$

This notation is almost but not quite standard: some authors do not include 0 in $\mathbb{N}$.

Often we use slight variants of the notation above: $\{3, 5, 7\}$ is the set consisting of $3, 5$, and 7; more generally, the set consisting of some list of elements is denoted by that list, enclosed in curly brackets, as in

$$\{n \mid n \in \mathbb{N} \text{ and } n \text{ is even}\} = \{0, 2, 4, \dots\}, \qquad 0.3.1$$

where again the vertical line | means “such that.”

The symbols are sometimes used backwards; for example, $A \supset B$ means $B \subset A$, as you probably guessed. Expressions are sometimes condensed:

$$\{x \in \mathbb{R} \mid x \text{ is a square}\} \quad \text{means} \quad \{x \mid x \in \mathbb{R} \text{ and } x \text{ is a square}\} \qquad 0.3.2$$

(i.e., the set of nonnegative real numbers).

A slightly more elaborate variation is *indexed unions and intersections*: if S_α is a collection of sets indexed by $\alpha \in A$, then

$$\bigcap_{\alpha \in A} S_\alpha \text{ denotes the intersection of all the } S_\alpha\text{, and}$$

$$\bigcup_{\alpha \in A} S_\alpha \text{ denotes their union.}$$

For instance, if $l_n \subset \mathbb{R}^2$ is the line of equation $y = n$, then $\bigcup_{n\in\mathbb{Z}} l_n$ is the set of points in the plane whose y-coordinate is an integer.

Although it may seem a bit pedantic, you should notice that

$$\bigcup_{n\in\mathbb{Z}} l_n \quad \text{and} \quad \{l_n \mid n \in \mathbb{Z}\}$$

are not the same thing: the first is a subset of the plane; an element of it is a point on one of the lines. The second is a set of lines, not a set of points. This is similar to one of the molehills which became mountains in the new-math days: telling the difference between ϕ and $\{\phi\}$, the set whose only element is the empty set.

We will use exponents to denote multiple products of sets; $A \times A \times \cdots \times A$ with n terms is denoted A^n: the set of n-tuples of elements of A. (The set of n-tuples of real numbers, $\mathbb{R}^n$, is central to this book; to a lesser extent, we will be interested in the set of n-tuples of complex numbers, $\mathbb{C}^n$.)

Russell’s paradox

In 1902, Bertrand Russell (1872–1970) wrote the logician Gottlob Frege a letter containing the following argument: Consider the set X of all sets that do not contain themselves. If $X \in X$, then X does contain itself, so $X \notin X$. But if $X \notin X$, then X is a set which does not contain itself, so $X \in X$.

“Your discovery of the contradiction caused me the greatest surprise and, I would almost say, consternation,” Frege replied, “since it has shaken the basis on which I intended to build arithmetic ... your discovery is very remarkable and will perhaps result in a great advance in logic, unwelcome as it may seem at first glance.”[1]

FIGURE 0.3.2.

Russell’s paradox has a long history. The Greeks knew it as the paradox of a barber living on the island of Milos, who decided to shave all the men of the island who did not shave themselves. Does the barber shave himself? Here the barber is Bertrand Russell. (Picture by Roger Hayward, provided by Pour la Science.)

As Figure 0.3.2 suggests, Russell’s paradox was (and remains) extremely perplexing. The “solution,” such as it is, is to say that the naive idea that any property defines a set is untenable, and that sets must be built up, allowing

[1]These letters by Russell and Frege are published in *From Frege to Gödel: A Source Book in Mathematical Logic, 1879–1931* (Harvard University Press, Cambridge, 1967), by Jean van Heijenoort, who in his youth was bodyguard to Leon Trotsky.

you to take subsets, unions, products, ... of sets already defined; moreover, to make the theory interesting, you must assume the existence of an infinite set. Set theory (still an active subject of research) consists of describing exactly the allowed construction procedures, and seeing what consequences can be derived.

Exercise for Section 0.3

0.3.1 Let E be a set, with subsets $A \subset E$ and $B \subset E$, and let $*$ be the operation $A * B = (E - A) \cap (E - B)$. Express the following sets using A, B, and $*$.

(a) $A \cup B$ (b) $A \cap B$ (c) $E - A$