MATH 1340 — Mathematics & Politics

Lecture 1 — June 22, 2015

Course Information

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- Office hours: M 1:00-3:00pm, Th 2:00-3:00pm
  in 112 Malott Hall (or by appointment)

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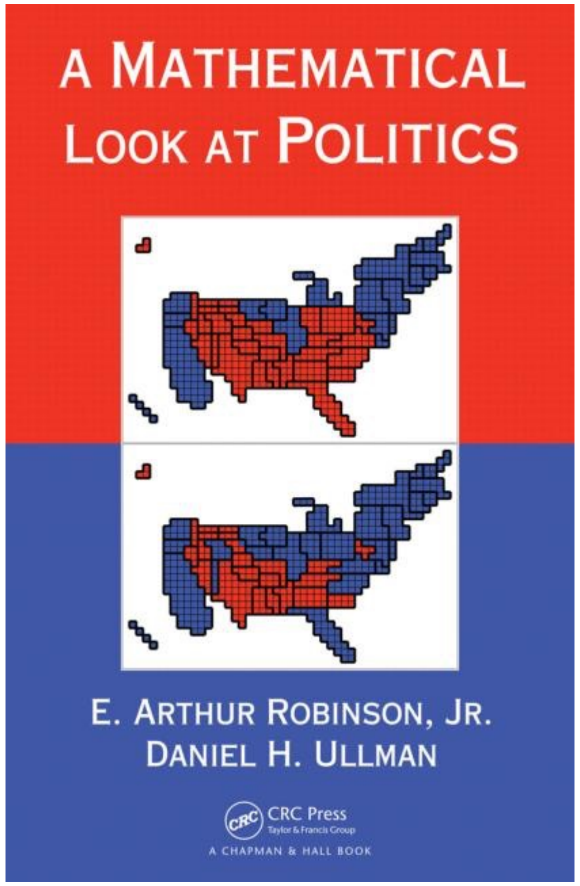
- Office hours: W 10:00am-11:00am in 218 Malott Hall

Course Information (cont'd)

- Lectures: MTWThF 11:30am-12:45pm
  224 Malott Hall

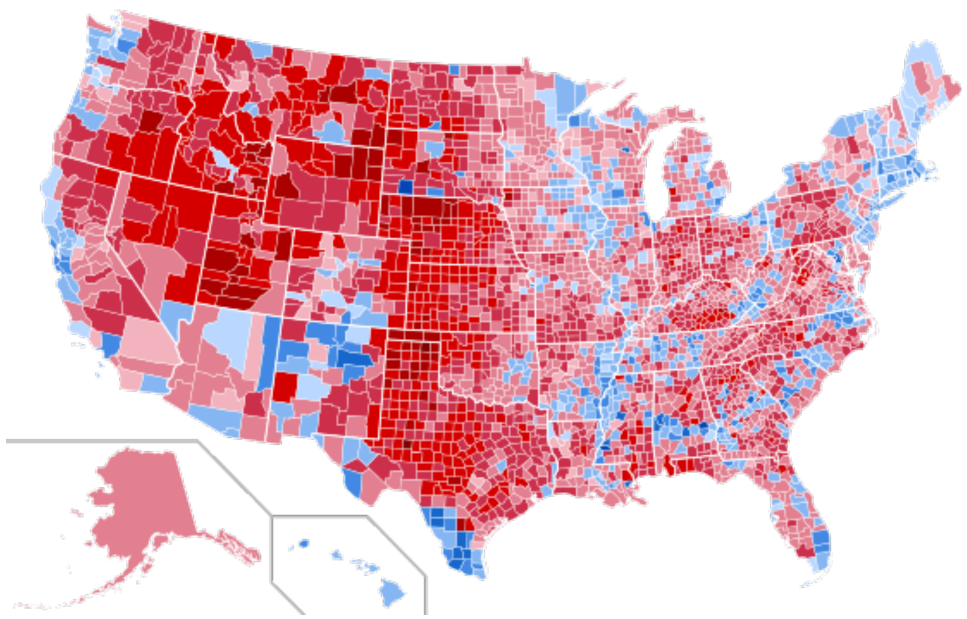
- Textbook (required): Robinson & Ullman,
  A Mathematical Look at Politics.
  Chapman & Hall, 2010.



- Website (for homework, slides, etc):
  www.math.cornell.edu/~ismythe/S15_1340.html

Course Outline

- Voting and Social Choice (Part I of R&U)
  - How do we choose a winning candidate in an election?
  - Voting methods ("social choice functions")
  - Two candidates?
  - Multiple candidates?
  - What characteristics do we want from a voting method?
  - Is it possible to have all of these characteristics? (Arrow's Theorem)



Course Outline

- Game Theory (Part III of R&U)
  - How do you maximize your payoff in a competitive game *knowing that your opponent is trying to do the same*?
  - Zero-sum and matrix games
  - Probabilistic ("mixed") strategies
  - Nash equilibria
  - The Prisoner's Dilemma

- Other topics (Parts II & IV of R&U, others)
  - Apportionment, the Electoral College, gerrymandering

Evaluation

- Homework (40%) will be assigned each Tuesday (due on Friday, in class), and each Friday (due the following Tuesday, in class). *The lowest two homework scores will be dropped.* See syllabus for more information.

- Test 1 (25%) on Voting and Social Choice (tentatively scheduled for July 7, in class).

- Test 2 (25%) on Game Theory (scheduled for August 3, at 1:30pm, in 224 Malott Hall).

- Class participation (10%). See syllabus for information.

Voting
and Social Choice

Electing a class president

- Suppose that we must elect a class president from a slate of two candidates.
- We need to chose a **voting method** or **social choice function**, that is, a method which selects a winning candidate (or candidates) given the votes of the class.
- For simplicity, we assume that everyone in the class *must* casts a vote by writing the name of *one* of the candidates on their ballot.
  - **No** blank ballots, write-ins, abstentions, etc.

Electing a class president (cont'd)

- What method should we use to pick the winner(s)? Why?
- Things that came up in class:
    - pick the candidate with the most votes; seems most likely to make the most people happy.
    - What if they get they same number of votes? Drop one of the the votes? Toss a coin? Have a tie?
    - What if one of the candidates decided that he wins, no matter what?
    - What if we weighted different students' votes differently?
    - We want a method that seems fair.

Voting and Social Choice

- What this section is **not** about:
  - who the candidates are, their policies, speeches, debates, etc
  - who the voters are, registration, suffrage, voter suppression, etc
  - how votes are cast, voting machines, etc
- What it is about:
  - once the votes are cast, how do we determine the winner(s)?

Aside on definitions

- Throughout this course, we will need to precise about our definitions. Thus, terms will be in **bold** when they are being precisely defined on these slides.
- When there is a (rare) conflict between a definition in the text and a definition in the slides, we will default to using the definition in the slides.
- Often, the definition of a term may clash with its common usage. Again, we will default to using the definition in the slides.

Ballots and profiles

- We begin with the case of two **candidates**, A and B.
  - One candidate elections are… not that interesting.
  - Candidates need not be people; they can be choices, yes or no, etc.
- We assume that all voters (the **electorate**), which we may number as 1, 2, 3,…, submit a **ballot** with either A or B on it (**not** both). The collection of all ballots is called a **profile**.

A	A	B	A	B

← Profile

Ballot of Voter 3

- We do **not** allow write-ins, abstentions, blank ballots, etc

Social choice functions — Two candidates

- A **function** is a rule that assigns to *every* possible input from one set (called the **domain**) a single output in another set (called the **codomain**)
    - Functions **cannot** be indifferent: they must *always* output something on a given input from the domain.
    - Functions **cannot** change their minds: if a function outputs “apple” on input “cow”, it must *always* output “apple” on input “cow”.
- A **social choice function** (or **voting method**) is a function with domain the set of *all possible profiles* from a fixed electorate, and codomain the set consisting of “A wins”, “B wins”, and “tie”.

(Non-)examples

- Suppose we were to pick a candidate by flipping a coin, say heads, A wins, and tails, B wins. Is this a social choice function, as we have defined them?
- Nope! This method could pick *different* winners for the *same* profile. These sorts of “probabilistic methods” are not allowed.
- How about the method that picks the candidate, A or B, with the most votes?
- If there is an *even number* of voters, this isn’t one either, because it doesn’t output anything if the candidates receive the same number of votes.

Simple-majority method

- The **simple-majority method** is the social choice function that, given a profile as input, outputs the candidate with the *most* number of votes, or “tie” if both candidates get the *same* number of votes.

Input

Output

A	A	B	A	B

A wins

A	B	B	A	A	B

Tie

Tabulated profiles

- Observe that the simple majority method depends only on the *number* of votes cast for each of the candidates, not who cast them.

- So, we can simplify the input to a **tabulated profile**, listing only the candidates and the number of votes.

Profile

A	A	B	A	B

Tabulated profile

A	B
3	2

- Can you think of situations where *who* casts the ballots should matter?

Super-majority methods

- Sometimes a bare majority is not regarded as sufficient; can you think of examples?

- Let p be a number with $1/2 < p \leq 1$. The **super-majority method** with **parameter** p is the social choice function that, given a profile as input, outputs the candidate whose fraction of the total vote is *at least* p, and "tie" otherwise.

  - If there are t voters, then the super-majority method selects as the winner the candidate with *at least* pt votes, and a tie otherwise.

  - The super-majority method also depends only on the tabulated profile.

  - When $p=1$, this method requires **unanimity** to produce a unique winner.

Quota methods

- A related method to the super-majority is the **quota method** with **quota** q (some whole number), which choses as the winner(s) the candidate(s) with at least q many votes.

- A super-majority method with parameter p is "functionally equivalent" (next time) to a quota method with quota $q =$ the smallest integer greater than pt.

- What happens if the quota is less than half of the size of the electorate in a two candidate election?

A silly method?

- The **simple-minority method** is the social choice function that, given a profile as input, outputs the candidate with the *least* number of votes, or “tie” if both candidates get the *same* number of votes.

- This may seem like a useless method, but never-the-less, it remains a valid social choice function.

Status quo methods

- In some cases, such as when voting Y or N on a piece of legislation, allowing ties makes no sense.

- We may designate one of the “candidates” as the **status quo**, and the other as the **challenger**, and use a social choice function (e.g., simple-majority, super-majority) to first determine if there is a *unique* winner, and if so, select that winner. If not, we select the status quo. This is a **status quo method.**

Status quo methods (cont'd)

- For example: When a bill is being considered in the US Senate, it requires a super-majority of 3/5 of all senators to end a filibuster.

- Considering Y (end the filibuster) and N (continue the filibuster) as our candidates, this describes a **super-majority with status quo** method, with parameter $p=3/5$, and status quo candidate N.

Status quo methods (cont'd)

- Since there are 100 senators, that means $100x3/5=60$ votes of Y are required to end the filibuster, otherwise, the filibuster (being the status quo) continues.

Y	N
57	43

Filibuster goes on

Y	N
60	40

Filibuster ends

Weighted voting method

- The **weighted voting method** is as follows: Suppose that there are n voters, labeled $1, 2, \ldots, n$ and voter i is given a positive number w_i of votes (called the **weight** of voter i). Let $t = w_1 + \ldots + w_n$ be the total number of votes. This method selects as winner(s) the candidate(s) who receive *more than half* (i.e., $t/2$) of the votes. If no such candidate exists, the result is a tie.

Weighted voting method (cont'd)

- For example: Suppose that there are five voters with weights $5, 5, 3, 2, 2$ respectively. Consider the following profiles (with voters listed in the above order). Note that we cannot use tabulated profiles in this method.

A	A	B	B	B

Who wins? A

A	B	A	B	B

Who wins? B

Functional equivalence

- Sometimes, two social choice functions with different descriptions may actually be, *for all intents and purposes*, the same.

- We say that two social choice functions are **functionally equivalent** if whenever they are given the same input profiles, they produce the same result.
  - A simple example is weighted voting with every voter getting weight = 1. This is functionally equivalent to the simple majority method.

- **Caution:** This notion is not in the text (explicitly), but we will use it repeatedly.

Functional equivalence (cont'd)

- For example: consider the weighted voting method with five voters having weights $12, 10, 9, 9, 8$. Note that $t=48$, so more than $t/2=24$ votes are required to win.

- We claim that this method is functionally equivalent to the simple-majority method. Why?

- From class: In the simple majority method, the winner is whoever gets 3 or more votes (no ties possible, since there are 5 voters).

- In this example, *every* combination of 3 votes has weight at least 26, since the smallest such combination is $9+9+8=26 > 24$. Thus, whoever receives *any* combination of 3 votes is the winner, and this is the only possible way to win, since combinations of 2 or fewer votes is at most $12+10=22 < 24$. Thus, this method is functionally equivalent to the simple majority method.

Other social choice functions

- We can combine the methods we've seen to create new social choice functions.

- Example 1.8 in R&U: Suppose that a small business has six partners, who vote on decisions with weights 10, 5, 5, 3, 2 and 1, respectively, and a decision requires a $p = 3/5$ super-majority to pass. Otherwise, the decision is rejected. Suppose that the partners are voting on a decision A, with alternative B being to reject the amendment. Does A pass with the following profile?

B	A	A	A	B	A

Other social choice functions (cont’d)

B	A	A	A	B	A

- Continuing the example: The total number of weighted votes is $t = 10+5+5+3+2+1 = 26$.
- At least $(3/5)x(26) = 15.6$ weighted votes are required to pass, that is, 16 weighted votes are required. (We can’t cut voters in half, and 15 weighted votes would not be enough.)
- Candidate A receives $5+5+3+1=14$ weighted votes, which is less than 16, and thus B (rejecting A), being the status quo, prevails.

Other social choice functions (cont’d)

- The **dictatorship method** is the social choice function in which a particular *voter* is chosen to be the **dictator**, and the winner is whichever candidate the dictator choses.
- The **monarchy method** is the social choice function in which a particular *candidate* is chosen to be the **monarch**, and is the winner regardless of the votes.
- The **parity method** is the social choice function which selects as the winner(s) the candidate(s) with an even number of votes, and a tie if no candidate does.
- **Caution:** This is far from an exhaustive list!

- Recommended reading: the syllabus, Sections 1.1-1.2 of R&U