MATH 1340 — Mathematics & Politics

Lecture 2 — June 23, 2015

An example

(Exercise 1.1 in R&U) Consider the following profile:

A	A	B	A	B	A	A	B	B	B	B	B	B	A	B	B	B

(1) What is the corresponding tabulated profile?

(2) Who wins under the simple majority method?

(3) the super-majority method with $p=2/3$?

(4) the super-majority method with $p=2/3$ but with A as the status quo?

(5) the weighted simple majority method where the first seven voters have 3 votes and the rest of the voters have 1?

An example (cont'd)

(1) The tabulated profile is

A	B
6	11

(2) B wins, having a majority of the votes.

(3) There is a tie, since both $6/17$ and $11/17$ (approx. 0.65) are less than $p=2/3$, so neither candidate has a $2/3$ super-majority.

(4) A wins, since the super-majority method does not produce a unique winner (see 3), and A is the status quo.

(5) A wins, since A receives 16 weighted votes, while B receives 15 weighted votes.

Criteria

- Which properties *should* a voting method have?
- What makes some voting methods better than others?
- Why does the simple majority method have such an obvious appeal?
- Note: The criteria we will focus on are those which seem desirable for electing public officials in a democracy; in other settings, these criteria may be less desirable.

Anonymity

- A social choice function is **anonymous** if the following holds:
  - Suppose we are given two profiles, say “before” and “after”,
  - and the “after” profile is the result of some voters in the “before” profile *exchanging* ballots amongst themselves.
  - Then, the outcome of the social choice function on these profiles must be the same.

- In other words, *rearranging the votes does not change the outcome*, so *all voters are treated equally*.

- For example, with an anonymous social choice function, the outcome of the following profiles must be the same:

before

A	B	A	A	B

after

B	A	B	A	A

Anonymity (cont'd)

Proposition: *Dictatorships are not anonymous.*

- Why?

- We need an example showing that, under a dictatorship, voters exchanging ballots *can* have an effect on the outcome of the election.

Anonymity (cont'd)

Proposition: *Dictatorships are not anonymous.*

- Why?

From class: Suppose that we have an electorate with 2 voters, and the first is the dictator. Consider the following two profiles:

before

A	B

after

B	A

Note that "after" is the result of the two voters exchanging ballots. But, these profiles have different outcomes: A wins in "before", while B wins in "after".

Thus, this method (a dictatorship) cannot be anonymous.

Proofs and counterexamples

- Throughout this part of the course, we will show that certain voting methods *fail* to satisfy certain criteria.

- In order to do so, all that is required is a **counterexample**, i.e., a single example (often a "before" and "after" profile) in which the relevant criteria is not met.

- Other times, we will need to show that criteria are met, or that certain facts about *all* voting methods are true.

- These demonstrations will require a **proof**, i.e., a step-by-step logical deduction of the desired property from the hypotheses. An example will **not** suffice.

Anonymity (cont'd)

Proposition: *A social choice function is anonymous if and only if its outcomes depend only on the tabulated profile.*

- This proposition is making two claims:
  - If a social choice function is anonymous, then its outcomes depend only on the tabulated profile ("the *only if* part")
  - If its outcomes depend only on the tabulated profile, then the social choice function is anonymous ("the *if* part").
- We must argue for both.

Anonymity (cont'd)

Proposition: *A social choice function is anonymous if and only if its outcomes depend only on the tabulated profile.*

Proof: (the "only if" part) Suppose that our social choice function is anonymous, and that we are given two profiles (which we may call "before" and "after") which correspond to the same tabulated profile. We must show that the outcome is the same for both profiles.

Since the two profiles have the same tabulated profile, that means that the same number of voters in both profiles voted for A, and likewise for B.

Thus, we can rearrange votes amongst the voters in "before" to obtain the "after" profile; if a particular voter voted for A in "before", but not in "after", then they can exchange with someone who voted for B in "before", but not in "after".

Since we can obtain "after" from "before" by exchanging votes, and our method is anonymous, the outcome must be the same.

Anonymity (cont'd)

Proposition: *A social choice function is anonymous if and only if its outcomes depend only on the tabulated profile.*

Proof: (the "if" part) Suppose that the outcome of our social choice function depends only on the tabulated profile, and that we are given two profiles "before" and "after" in which the latter is the result of voters exchanging votes from the former. We must show that the outcome is the same.

Since rearranging the votes does not change the total number of votes for A, or the total number of votes for B, the two profiles must have the same tabulated profile.

Thus, by assumption, the outcome is the same, showing that the method is anonymous.

QED

Anonymity (cont'd)

- We have already seen that the simple majority and super-majority methods (with or without status quo) depend only on the tabulated profile, and thus are anonymous.

- Weighted voting methods, in general, are not anonymous. Can you come up with a counterexample?

- **Caution:** *Some* weighted voting methods are anonymous (e.g., when each voter gets a weight of *1*), but because some are not, we must conclude that weighted voting methods, in general, are not.

Note on terminology

- R&U use the term “winner” to usually mean *unique winner* (i.e., not a tie).

- I will try to always specify *unique winner* in class and the slides.

Neutrality

- A social choice function is **neutral** if the following holds:
  - Suppose that we are given two profiles, “before” and “after”,
  - and the “after” profile is the result of *all* of the voters for A (in “before”) changing their votes to B, and *all* of the voters for B changing their votes to A.
  - Then, the unique winner must change from A to B, or from B to A, or the outcome remains a tie.

- In other words, *swapping the candidates swaps the winners,* so *all candidates are treated equally.*

- For example, if we are using a neutral social choice function, and A (B) wins in the before profile below, then B (A) must win in the after profile:

before

A	B	A	A	B

after

B	A	B	B	A

Neutrality (cont'd)

Proposition: *Monarchies are not neutral.*

- Why?

From class: Suppose that candidate A is the monarch, and we are given the following profiles (actually, any profiles will work, but an example suffices):

before

A	A

after

B	B

A is the unique winner in "before", and remains so in "after", even though the A votes have been replaced by B votes.

Thus, this method is not neutral.

Neutrality (cont'd)

Proposition: *The simple majority and super-majority methods are neutral.*

- Why?

Proof (from class): (We'll do the case for simple majority; super-majority is similar.)
Suppose that we are using the simple majority method, and there are two profiles, "before" and "after" in which the A votes and B votes have been swapped, as in the definition of neutrality.

Case 1: A is the unique winner in "before". Since we are using the simple majority method, A has more votes than B in "before". Since all of the A votes have become B votes in "after" (and vice-versa), B must now have more votes than A in "after". Thus, B is the unique winner in "after".

Case 2: B is the unique winner in "before". This argument is exactly the same as Case 1, with the roles of A and B reversed.

Case 3: There is a tie. Then, A and B have the same number of votes in "before", and this must remain true in "after". Thus, A and B remain tied in "after".

QED

Neutrality (cont'd)

Proposition: *The simple-majority (or super-majority) with status quo method is not neutral.*

- Try it! (For the simple-majority with status quo case, this is on the homework, Problem 1.8(b).)

Monotonicity

- A social choice function is **monotone** if the following holds:
  - Suppose that we are given two profiles, “before” and “after”, in which a candidate, say X, is the *unique* winner in the “before” profile,
  - and the “after” profile is obtained from the “before” profile by one or more voters changing their vote from the *other* candidate to X.
  - Then, X must remain the unique winner.

- Monotonicity ensures that *more* votes for candidate will not hurt their chances of winning.

Monotonicity

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  - Suppose that we are given two profiles, “before” and “after”, in which a candidate, say X, is the *unique* winner in the “before” profile,
  - and the “after” profile is obtained from the “before” profile by one or more voters changing their vote from the *other* candidate to X.
  - Then, X must remain the unique winner.

- For example: if we are using a monotone social choice function and B wins in the “before” profile, B must still win in the “after” profile:

before

A	B	B	A	B	B	B	A	A	A

after

A	B	B	A	B	B	B	A	B	A

Monotonicity (cont'd)

Proposition: *The simple majority and super-majority methods are monotone.*

- Why?

Proof (from class): (The simple majority case is very similar, try thinking through it.)
Suppose that our method is the super-majority method with parameter p, where $1/2 < p < 1$. Suppose that we are given two profiles "before" and "after" in which a candidate, say A, is the unique winner in "before", and one or more voters have changed their vote from B to A in "after". We need to show that A remains the unique winner in "after".

Since A was the unique winner in "before" and we are using the super-majority method with parameter p, she must have received at least pt many votes in "before", where t is the number of voters. But then, A receives at least $pt+1 > pt$ many votes in "after", and thus must still be the unique winner.

Similarly (or by neutrality), the same is true if candidate B was the unique winner in "before".

QED

Monotonicity (cont'd)

- It's also easy to see that the dictatorship and monarchy methods are monotone. (Try this!)

- Have we seen a method which fails this?

Proposition: *The parity method is not monotone.*

- Why?

- From class: Consider the profiles (fill in the details)

before

B	B	B	B	A	B	B	A	B

after

B	B	B	B	A	B	B	A	A

Decisiveness

- A social choice function is **decisive** if it always chooses a *unique* winner, i.e., never results in a tie.

- A social choice function is **nearly decisive** if the only situation in which a tie can occur, is if both candidates receive the same number of votes.

Proposition: *The simple-majority method is nearly decisive, but may fail to be decisive.*

- Why? (More on this next time, or on p. 18 of R&U)

Decisiveness (cont'd)

Proposition: *The status quo, dictatorship and monarchy methods are all decisive.*

- Why?
- Problem? Status quo and monarchy are not neutral, and dictatorship is not anonymous.

Criteria roundup

- Here's a chart (from p. 19 in R&U) listing which criteria some of the voting methods satisfy and fail to satisfy:

	Anonymous	Neutral	Monotone	Decisive	Nearly decisive
Simple majority	yes	yes	yes	no	yes
Super-majority	yes	yes	yes	no	no
(Simple majority w/) Status quo	yes	no	yes	yes	yes
Weighted	no	yes	yes	no	no
Parity	yes	yes	no	no	no
Dictator	no	yes	yes	yes	yes
Monarchy	yes	no	yes	yes	yes
All ties	yes	yes	yes	no	no

- You should try to convince yourself of the facts in this chart which we haven't already shown today.

The Big Question

- Can we find a social choice function that is anonymous, neutral, monotone, *and decisive*?

- Recommended reading: Sections 1.3-1.4 of R&U
- Problem set #1 (due in class on Friday) has been posted on the course website. (Note: Problem 1.4 from the text requires material from next class.)