MATH 1340 — Mathematics & Politics

Lecture 3 — June 24, 2015

Decisiveness

- Recall from last time:

- A social choice function is **decisive** if it always chooses a *unique* winner, and **nearly decisive** if the *only* situation in which a tie can occur, is if both candidates receive the same number of votes.

Proposition: *The simple majority method is nearly decisive, but may fail to be decisive.*

- Why?

Decisiveness (cont'd)

Proposition: *The simple majority method is nearly decisive, but may fail to be decisive.*

Proof: The simple majority method fails to be decisive because it yields a tie when both candidates receive the same number of votes (which is possible with any even-sized electorate). Why is it nearly decisive? That is, why is a tie *only* possible when the candidates receive the same number of votes?

Consider the two cases: Either one candidate receives more votes than the other, in which case that candidate is the unique winner, or both candidates receive the same number of votes and there is a tie. Since these are the only possibilities, we can conclude that the method is nearly decisive.

QED

The Big Question(s)

- Thus, the simple majority method is anonymous, neutral, monotone, and nearly decisive.
- Are there other methods that satisfy all of these?
- Can we find a social choice function that is anonymous, neutral, monotone, *and decisive*?

May’s Theorem

- In 1952, mathematician Kenneth May proved that the *only* voting method for two candidates which is anonymous, neutral, monotone and nearly decisive is the simple majority method. That is:

Kenneth May (1915-1977) in 1969

May’s Theorem: *In an election with two candidates, a social choice function that is anonymous, neutral, monotone, and nearly decisive is (functionally equivalent to) the simple majority method.*

May's Theorem (cont'd)

<u>May's Theorem:</u> *In an election with two candidates, a social choice function that is anonymous, neutral, monotone, and nearly decisive is (functionally equivalent to) the simple majority method.*

- In other words, if we have *any* method which satisfies anonymity, neutrality, monotonicity and is nearly decisive, then it *is* the simple majority method in disguise.
- Thus, the simple majority method is the *only* voting method having all of these properties, not just of the ones we have studied, but of *all theoretically possible* voting methods!

An impossibility result

- Moreover, it tells us that our search for a *decisive* method satisfying these criteria is futile, since the simple majority method is not decisive (if the electorate is even). That is:

Corollary: *It is impossible for a social choice function with two candidates (and an even-sized electorate) to satisfy anonymity, neutrality, monotonicity and decisiveness.*

May’s Theorem (cont’d)

- Let’s prove May’s Theorem.

- A word about **proofs by contradiction**: Often, in order to prove that some statement “___” is false, we will suppose that it is true, and then derive a contradiction. As long as our reasoning is valid, this contradiction tells us that our assumption, namely “___”, is false.

- We will use this technique a few times during this proof.

Proof of May's Theorem

Proof (from class): Suppose that our social choice function is anonymous, neutral, monotone and nearly decisive. We must show that our method is functionally equivalent to the simple majority method, that is, the candidate with the most votes wins, and if both candidates receive the same number of votes, there is a tie. Observe that by anonymity, we need only consider tabulated profiles (i.e., the number of votes a candidate gets is all that matters).

Suppose we are given a profile. Let a be the number of votes for candidate A, and b the number of votes for candidate B, so that $t=a+b$ is the total number of votes. We consider two cases:

Case 1: t is even.

If $a=b=t/2$, then we claim that there must be a tie. Why? If not, then there is a unique winner, so if we create a new profile by switching all A votes to B votes and vice-versa, we must switch winners, by neutrality. However, this new profile has the same tabulated profile, so the winner must be the same, by anonymity. This contradiction shows that there cannot be a unique winner, hence there is a tie.

Proof of May's Theorem (cont'd)

(Still considering t even)

Next suppose that A has a majority of that votes, that is, $a \geq t/2+1$. We need to show that A must be the unique winner. By near decisiveness, we cannot have a tie in this scenario, so it remains to show that B cannot be the unique winner.

Suppose that B was the unique winner, with $b = t\text{-}a \leq t\text{-}(t/2+1) = t/2\text{-}1 < t/2$ votes. By monotonicity, B would also be the unique winner if he had $t/2$ votes, but we have already seen (on the previous slide) that this case results in a tie. This contradiction shows that B cannot be the unique winner.

Since there must be some outcome, it must be the case that A is the unique winner, as desired.

The same holds if B has a majority of vote, by neutrality. Thus, provided t is even, our method is functionally equivalent to the simple majority method.

Proof of May's Theorem (cont'd)

Case 1: t is odd.

In this case, it's impossible for A and B to have the same number of votes (since $t/2$ is not a whole number), so by near decisiveness, there cannot be a tie.

Suppose that A has $a = t/2+1/2$ many votes, the smallest possible majority. We want to show that A must be the unique winner (then monotonicity will imply that A will be the unique winner with more votes as well). Note that B gets $b = t\text{-}a = t\text{-}(t/2+1/2) = t/2\text{-}1/2 = a\text{-}1$ many votes. We've already seen that there cannot be a tie.

Suppose that B was the unique winner, with his $a\text{-}1$ many votes. By neutrality, A would also win with $a\text{-}1$ votes, and by monotonicity, with a many votes. But that is exactly the scenario in which B has $a\text{-}1$ votes, so this is a contradiction.

Thus B cannot be the unique winner, so A must the unique winner. So, A is the unique winner if she gets the majority of the vote. Again, by neutrality, the same would hold of B, showing that our method is functionally equivalent to the simple majority method.

QED

A generalization of May’s Theorem

- By imitating the proof of May’s Theorem, we can also obtain the following related result. Recall that the **all ties method** is the social choice function that declares a tie regardless of the input profile.

Generalized May’s Theorem: *In an election with two candidates, a voting method that is anonymous, neutral and monotone must be (functionally equivalent to) either the simple majority method, super-majority method, or the all ties method.*

An example

- Example (Exercise 1.3 in R&U): Suppose you favor one (and only one) of two alternatives but only 10% of the electorate agrees with your position. Is there a voting method that leads to victory for your position that is:

  A. Anonymous?

  B. Anonymous and neutral?

  C. Anonymous, neutral and monotone?

An example (cont'd)

A. Yes, for example, the monarchy method with your preference as the "monarch" candidate.

B. Yes, for example, the simple *minority* method (winner is candidate with *least* number of votes, otherwise tie).

C. No, the Generalized May's Theorem tells us that the only such methods are simple majority, super majority or all ties, and none of these will let your preference win in this scenario.

Voting and Social Choice

Multiple candidates

An example

- Suppose there are 3 candidates, B, G and N, in an election with… 5,922,531 voters.

- The *tabulated preferences* of the voters are as follows:

1,893,313	1,019,477	2,329,802	582,451	97,488
B	B	G	G	N
N	G	N	B	G
G	N	B	N	B

- Note: the sum of the 1st and 2nd columns is 2,912,790, and the sum of the 3rd and 4th columns is 2,912,253.

- Who should be declared the winner?

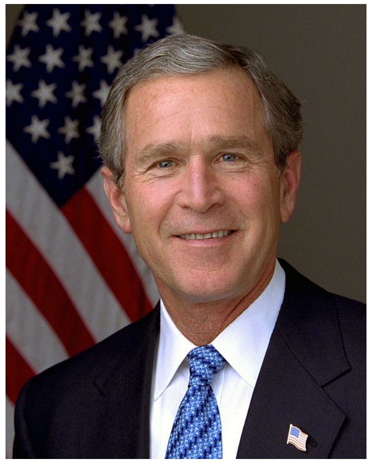
An example (cont'd)

1,893,313	1,019,477	2,329,802	582,451	97,488
B	B	G	G	N
N	G	N	B	G
G	N	B	N	B

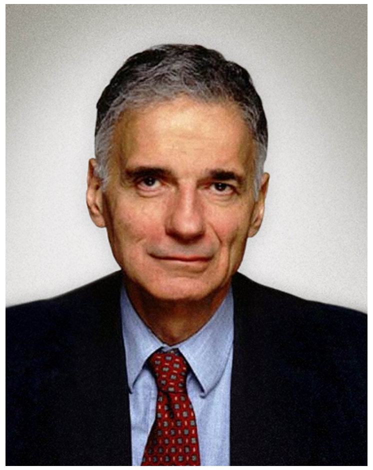
- Observe that candidate B receives the *plurality* of first place votes, but not a majority.
- However, a majority of voters (3,009,741) *prefer* (that is, rank higher) candidate G over candidate B.

An example (cont'd)

- This fictional example is based on the outcome of the 2000 US Presidential Election in the state of Florida (we have invented the preference orders, and removed other minor candidates, but the first place totals are factual)


George W. Bush (R)


Albert Gore Jr. (D)


Ralph Nader (G)

- Bush received a plurality of votes, resulting in Florida (and thus the country) selecting Bush as the winner.

- However, it is reasonable to assume that Nader's supporters preferred Gore to Bush, meaning a majority preferred Gore to Bush.

- This "plurality method" cannot reflect this fact.

A second example

- Consider the tabulated preferences below, based (from real data and an example in T&P) on the results of the 1980 US Senate race in New York between Alfonse D’Amato (R), Elizabeth Holtzman (D), and Jacob Javits (I).

1,319,830	1,379,822	892,725	1,725,936	422,891	241,653
D’Amato	D’Amato	Holtzman	Holtzman	Javits	Javits
Holtzman	Javits	D’Amato	Javits	Holtzman	D’Amato
Javits	Holtzman	Javits	D’Amato	D’Amato	Holtzman

- D’Amato receive a plurality (45%) of the vote and won, while Holtzman and Javits received 44% and 11% respectively.

- Again, a majority (51%) preferred Holtzman to D’Amato.

A second example (cont'd)

1,319,830	1,379,822	892,725	1,725,936	422,891	241,653
D'Amato	D'Amato	Holtzman	Holtzman	Javits	Javits
Holtzman	Javits	D'Amato	Javits	Holtzman	D'Amato
Javits	Holtzman	Javits	D'Amato	D'Amato	Holtzman

- However, there is more:
- Notice that in a one-on-one challenge Holtzman would defeat Javits nearly 2 to 1 (3,938,491 to 2,044,366): we see this by removing D'Amato in the above, and tallying up the new first-place votes for Holtzman and Javits.
- Moreover, in a one-on-one challenge, Holtzman would *also* defeat D'Amato, (3,041,552 to 2,941,305) (!)

Social choice and multiple candidates

- These examples highlight some of the difficulties that we encounter in elections with more than 2 candidates.

- We will now consider such elections, with an arbitrary (finite) set of **candidates** (called the **slate**), usually A, B, C,…

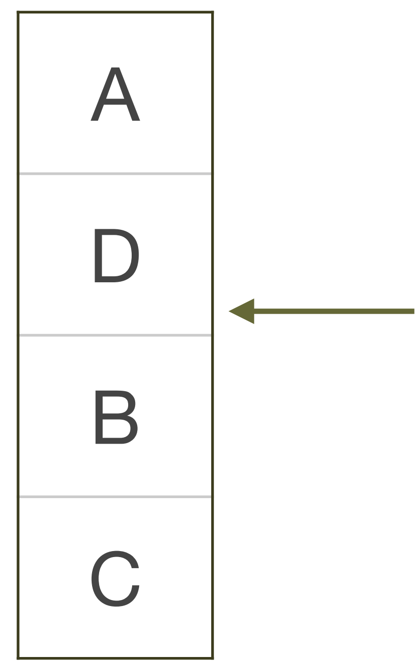
  - Again, candidates need not be people, but we will focus on examples of electing public officials

  - We will assume that there are *2 or more* voters, and *2 or more* candidates

- An immediate question: What sort of ballot should be used?

Preference ballots

- We will assume that each voter will submit a **preference ballot**, on which he or she lists the candidate in *descending order of preference* (their **preference order**).



preference ballot with
A as first choice,
D as second choice
B as third choice,
and C as fourth choice

  - Every voter *must* rank *all* of the candidates, **cannot** express equal preference or indifference among candidates, and **no** write-ins, blank ballots, etc.
  - This may be slightly unrealistic, but it is an idealized situation (or a **mathematical model**) which allows us to develop a richer theory

Preference ballots (cont’d)

- We will also assume that voters are **rational** in the following sense: if a voter prefers candidate A to B and B to C, then she must also prefer to A to C. That is, she holds **transitive** preferences.

- Note: we can still consider elections which use the usual “vote-for-one” ballots simply by ignoring lower ordered preferences.

- Our ballots for two-candidate elections can be thought of as preference ballots.

Profiles

- The collection of all preference ballots cast by a fixed a electorate is called a **profile**.

A	B	A	C	A	B	C
B	C	B	A	B	C	A
C	A	C	B	C	A	B

← Profile

↖ Preference ballot of Voter 3

Social choice functions — Multiple candidates

- A **social choice function** (or **voting method**) is a function with domain the set of all possible profiles from a fixed electorate, and codomain *every nonempty subset of the slate of candidates.*
  - The candidates in the resulting output set are called the **winners**, while the candidates not in that set are the **losers**.
  - We ensure that the social choice function must output *some* winner by only allowing *nonempty* sets of candidates.
  - We allow all possible combinations of candidates as ties.
- For example: If the candidates are A, B and C, what are the possible outputs of a social choice function?

Social choice functions (cont'd)

- An aside: How many social choice functions are there?
- Consider the case of 3 candidates, and only 4 voters. There are 6 possible preference ballots:

A	A	B	B	C	C
B	C	A	C	A	B
C	B	C	A	B	A

  - Since each voter must chose a preference ballot, there are $6^4 = 1296$ possible input profiles.
  - For each of those 1296 profiles, a social choice function must chose one of 7 possible sets of winners.
  - There are 7^{1296} (a number with 1096 digits!) ways to do this…

- Recommended reading: Sections 1.5 and 2.1 in R&U.
- Optional exercise: Try to prove the Generalized May's Theorem by replicating the proof of May's Theorem.
- Optional reading: May's original paper*: "A Set of Independent Necessary and Sufficient Conditions for Simple Majority Decision", *Econometrica,* Vol. 20, No. 4 (Oct., 1952), pp. 680-684.

*If you are not on a Cornell network, you can use this link instead and access the paper with your NetID. Adding ".proxy.library.cornell.edu" to the first part of a URL behind a paywall works for most scientific journals and other publications (the *New York Times*, etc).