MATH 1340 — Mathematics & Politics

Lecture 4 — June 25, 2015

Profiles and social choice functions

- Recall from last time: in an election with a slate of candidates A, B, C,…, voters submit a **preference ballot**, so that a possible **profile** may look like:

A	B	A	C	B
B	C	B	A	A
C	A	C	B	C

- A **social choice function** (or **voting method**) is a function with domain the set of all possible profiles from a fixed electorate, and codomain *every nonempty subset of the slate of candidates.*

Plurality method

- The **plurality method** is the social choice function that selects as the winners the candidates who are ranked as the *first choice* by the *most number* (**plurality**) of voters.

A	B	A	C	A
B	C	B	A	C
C	A	C	B	B

→ **A wins**

A	B	A	C	B
B	C	B	A	A
C	A	C	B	C

→ **A & B win**

Majority?

- Note that we did not require a *majority* of first-place votes for a candidate to win in this method, and we cannot do so; it could result in no winner being picked, violating that social choice functions can only output *nonempty* sets.

- Find an example of a profile in which there is a unique winner with the plurality method, but no candidate receives a majority of first-place votes.

Tabulated profiles

- Observe that the plurality method only depends on the *number* of voters having a particular preference order, not who cast them.

- So again, we can simplify the input to a **tabulated profile**, listing only the number of voters having each preference order.

Profile

A	B	A	C	A	B	C
B	C	B	A	B	C	A
C	A	C	B	C	A	B

Tabulated profile

3	2	2
A	B	C
B	C	A
C	A	B

Alternatives to plurality?

- Consider the following tabulated profile:

5	4	4	4	3
A	B	C	D	E
B	C	B	B	D
C	E	D	E	B
E	D	E	C	C
D	A	A	A	A

- The plurality method selects A as the winner, however, A is considered to be the worst candidate by 15/20 (75%) of the electorate, all of whom prefer candidate B.

- Are there reasonable methods which might select B instead?

Borda count method

- The **Borda count method** (due to Jean Charles de Borda, 1781) is the social choice function described as follows:
  - If there are n candidates in the slate, then assign n-1 points to a candidate for every voter who ranks them first, n-2 points for every voter who ranks them second, and so forth, until you assign 1 point for every voter who ranks them in position n-1 (i.e., second to last), and assign no points for last place votes.
  - Tally the points, and the winners are the candidates who receive the most points.

Jean Charles de Borda
(1733-1799)

Borda count method (cont'd)

- Let's work out the Borda count method on the profile

5	4	4	4	3
A	B	C	D	E
B	C	B	B	D
C	E	D	E	B
E	D	E	C	C
D	A	A	A	A

- There are $n=5$ candidates, so first place is 4 points, second place 3 points, third place 2 points, fourth place 1 point, and fifth (last) place none.

- A gets 5x(4 points) (for the first column), and no others (last place is worth 0 points), so A gets 20 points.

- B gets 5x(3 points) (first column) + 4x(4 points) (second column) + 4x(3 points) (third column) + 4x(3 points) (fourth column) + 3x(2 points) (last column) = 61 points.

Borda count method (cont'd)

- A gets *20* points.
- B gets *61* points.
- C gets *5x2 + 4x3 + 4x4 + 4x1 + 3x1 = 45* points
- D gets *5x0 + 4x1 + 4x2 + 4x4 + 3x3 = 37* points
- E gets *5x1 + 4x2 + 4x1 + 4x2 + 3x4 = 37* points
- Thus, B is the winner in the Borda count method.

5	4	4	4	3
A	B	C	D	E
B	C	B	B	D
C	E	D	E	B
E	D	E	C	C
D	A	A	A	A

Borda count method (cont'd)

- The Borda count method is not without its flaws. Consider the following profile:

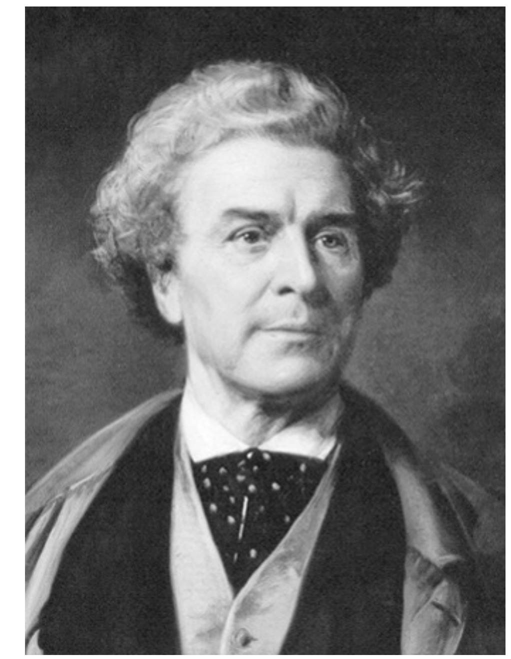
5	4	4	4	3
B	C	A	D	E
C	A	B	A	A
E	B	E	B	B
D	E	D	E	D
A	D	C	C	C

- One can check (try it!) that the Borda count totals for candidates A, B, C, D, and E are 49, 54, 31, 28 and 38, respectively, so candidate B would win.

- However, 75% of the electorate prefers candidate A to B.

Hare's method and instant run-off

- **Hare's method** (due to Thomas Hare, 1861), or the **instant run-off method**, is the social choice function described as follows:
  - Tally the first-place votes. If a candidate receives a *majority* of first-place votes, declare them the winner.
  - Otherwise, identify the candidates with the *fewest first-place votes* and eliminate them from consideration, moving up all of lower ranked candidates on each voter's preference list, yielding a new profile with the remaining candidates.
  - Repeat this process until there is a majority winner, who is thus declared the winner, or until there is an exact tie among all remaining candidates, in which case those remaining candidates are all declared winners.



Thomas Hare
(1806-1891)

Hare's method (cont'd)

- Let's work out Hare's method on the profile (recall B wins with the Borda count method)

5	4	4	4	3
B	C	A	D	E
C	A	B	A	A
E	B	E	B	B
D	E	D	E	D
A	D	C	C	C

- Is there a majority winner? No. Who has the least first-place votes? E.
- Eliminate E to get a new profile:

5	4	4	4	3
B	C	A	D	A
C	A	B	A	B
D	B	D	B	D
A	D	C	C	C

Hare's method (cont'd)

- Start over with the new profile.
- Is there a majority winner? No.
  Who has the least first-place votes? C and D.

5	4	4	4	3
B	C	A	D	A
C	A	B	A	B
D	B	D	B	D
A	D	C	C	C

- Eliminate C and D to get a new profile:

5	4	4	4	3
B	A	A	A	A
A	B	B	B	B

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5	15
B	A
A	B

- Is there a majority winner? Yes, A.
- Thus, A is the winner in Hare's method. (Recall that B was the winner with the Borda count, so these are genuinely different methods.)

Hare’s method (cont’d)

- While we described Hare’s method as consisting of rounds, the original profile contains all of the information needed to compute the winners, which can be done instantly by a computer.

- This form of run-off election assumes that voters hold consistent preferences once candidates are removed from consideration (unlike in run-off elections where voters must return to the polls after each round).

- Hare’s method (or variations) is used for some municipal elections in the US (including San Francisco and Oakland, CA, and Burlington, VT), for electing leaders of political parties in Canada and the UK, and more widely in Australia, Ireland and India.

Coombs' method

- **Coombs' method** (due to Clyde Coombs) is the social choice function that operates exactly like Hare's method, except instead of eliminating at each stage the candidate with the least first-place votes, we eliminate the candidate with the *most last-place votes*.

Clyde Coombs
(1912-1988)

- **Caution:** The text is inconsistent in its use of Coombs' method; we will *always* assume you check if there is a majority winner *first*, then do the elimination.

- Can you come up with a profile that yields different results for the Hare and Coombs' methods? (say, 3 candidates)

Coombs' method (cont'd)

- Consider the following tabulated profile:

2	2	1
A	B	C
C	C	A
B	A	B

- Apply Hare's method: There is no majority winner, so we eliminate C (having only 1 first place vote) and get the new profile:

3	2
A	B
B	A

- A is the majority winner now, so A wins in Hare's method.
- Applying Coombs' method: there is no majority winner in the original profile, so we eliminate B (having 3 last place votes) and get:

2	3
A	C
C	A

- C is the majority winner now, so C wins in Coombs' method.
- Thus, Hare's and Coombs' methods are different.

Hare’s and Coombs’ methods

- Hare’s method eliminates candidates who are “least loved”
- Coombs’ method eliminates candidates who are “most hated”, so “bland” candidates are more likely to survive.
- Both methods have dire consequences for candidates eliminated early, regardless of how they would perform in head-to-head match-ups.

Copeland's method

- **Copeland's method** (due to A. H. Copeland, 1951) is the social choice function described as follows:
  - Each candidate is awarded 1 point for every other candidate that he or she can beat in a head-to-head match-up (using the simple majority method).
  - Each candidate is awarded $1/2$ point for every other candidate that he or she ties in a head-to-head match-up.
  - The candidates with the most points are the winners.

Copeland’s method (cont’d)

- Let’s work out Copeland’s method on the profile

A	E	A	D	B
C	C	B	E	D
B	A	C	C	C
D	B	D	A	E
E	D	E	B	A

- We need to look at each possible pairing, like a “round robin” tournament.

Copeland’s method (cont’d)

- Consider all possible head-to-head match-ups by eliminating all but two candidates.

A	E	A	D	B
C	C	B	E	D
B	A	C	C	C
D	B	D	A	E
E	D	E	B	A

4	1
A	B
B	A

2	3
A	C
C	A

3	2
A	D
D	A

2	3
A	E
E	A

2	3
B	C
C	B

4	1
B	D
D	B

3	2
B	E
E	B

3	2
C	D
D	C

3	2
C	E
E	C

3	2
D	E
E	D

- A and B both get 2 points, C gets 4 points, D and E both get 1 point. Thus, C is the winner in Copeland’s method.

Tabulated profiles

- Every method (plurality, Borda count, Hare, Coombs, and Copeland) that we have seen so far depends only on the tabulated profile.

- When we discuss criteria (starting tomorrow) for social choice functions, we will see that this corresponds once again to **anonymity**.

Familiar methods

- Let v be one of the voters. The **dictatorship method** with **dictator** v is the social choice function in which the candidate ranked first by v is selected as the unique winner.

- Let X be one of the candidates. The **monarchy method** with **monarch** X is the social choice function which selects X as the unique winner, for any profile.

- The **all-ties method** is the social choice function that selects *all* of the candidates to be winners, on any profile.

- Recommended reading: Sections 2.2-2.3 in R&U

- Optional reading: To see what happens when political parties (like both the Democrats and Republicans) don’t use an instant run-off method for their presidential nominating conventions, look up the 1924 Democratic National Convention, and the term “brokered convention”.