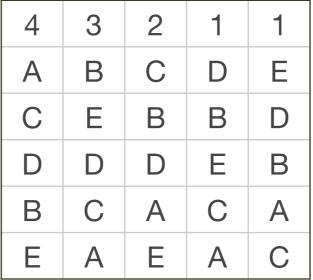
MATH 1340 — Mathematics & Politics

Lecture 5 — June 26, 2015

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An example

 (Exercise 2.1 in R&U) Consider the following profile involving 5 candidates and 11 voters:



- (a) Who wins the election if the plurality method is used?
- (b) Who wins the election if the Borda count method is used?
- (c) Who wins the election if the Coombs' method is used?

An example (cont'd)

- (a) A wins with a plurality of 4 firstplace votes.
- (b) With Borda count:

A gets 4x4+0+2x1+0+1x1 = 19 points B gets 4x1+3x4+2x3+1x3+1x2 = 27 points C gets 4x3+3x1+2x4+1x1+0 = 24 points D gets 4x2+3x2+2x2+1x4+1x3 = 25 points E gets 0+3x3+0+1x2+1x4 = 15 points, so B wins.

(c) With Coombs' method: There is no majority first-place winner, so we eliminate E (having the most last-place votes), and get a new profile.

There is still no majority winner, so we eliminate A, and get a new profile

C is now the majority winner, thus C wins.

4	3	2	1	1
A	В	С	D	Е
С	Е	В	В	D
D	D	D	Е	В
В	С	А	С	А
E	А	Е	А	С

_					
	4	3	2	1	1
	Α	В	С	D	D
	С	D	В	В	В
	D	С	D	С	А
	В	А	А	А	С
_					

4	3	2	2
С	В	С	D
D	D	В	В
В	С	D	С

Positional methods

- The following methods generalize the Borda count:
- For *n* candidates, the **positional method** associated to the sequence of numbers $a_1 \ge a_2 \ge ... \ge a_n$ is the social choice function described as follows:
 - Assign, for each candidate, *a*₁ points to every first-place vote, *a*₂ points to every second-place vote, and so on, until we assign *a*_n points for every last-place vote.
 - Tally the points, and the winners are the candidates who receive the most points.
- We denote this method by $P(a_1, a_{2,...,a_n})$.

Positional methods (cont'd)

- Observe that the Borda count, for n candidates, is exactly the positional method P(n-1,n-2,...,1,0).
- The plurality method can be thought of as the positional method P(1,0,0,...,0).
- The **antiplurality method**, which picks as winners those with the *fewest last-place votes*, can be thought of as the positional method *P*(*1*,*1*,...,*1*,*0*).

Functional equivalence

- We say that two social choice functions are functionally equivalent if whenever they are given the same input profiles, they produce the same result.
- Again, this notion is **not** explicit in the text, but will be used repeatedly.

Criteria

- We again need to specify **criteria** (properties which may or may not hold of a given social choice function) in order to determine which methods are acceptable, and which are not.
- Some of these will be familiar from the two-candidate case, others will be quite new.

Anonymity

- A social choice function is **anonymous** if the following holds:
 - Suppose we are given two profiles, "before" and "after",
 - the "after" the result of some voters in the "before" exchanging their preference ballots.
 - Then, the outcome of the social choice function on these profiles must be the same.
- As in the two-candidate case, anonymity means that rearranging the ballots does not change the outcome, so all voters are treated equally.

Anonymity (cont'd)

As in the two-candidate case we have (with a similar proof):

<u>Proposition:</u> A social choice function is anonymous if and only if it depends only on the tabulated profile.

• You can try to write out a detailed proof yourself.

<u>Corollary:</u> Plurality, Borda count, Hare, Coombs, Copeland, and all positional methods are anonymous.

• The dictatorship method is, of course, not anonymous.

Neutrality

- A social choice function is **neutral** if the following holds:
 - Suppose that we are given two profiles, "before" and "after". In "before", candidate X is a winner, and
 - for some other candidate Y, all of the voters have interchanged X with Y in their preference orders in "after".
 - Then, the function must select Y as a winner in "after".
- Again, this means that candidates are treated equally; if support for one candidate can allow that candidate to win, then the same support can allow any other candidate to win.

<u>Proposition:</u> Plurality, Borda count, Hare, Coombs, Copeland, and all positional methods all neutral.

- Why? (See next slides)
- Monarchy is, of course, not neutral.

• Positional methods (thus plurality and Borda count, as well) are neutral:

Suppose we are given profiles "before" and "after" in which X is a winner with the positional method $P(a_1, a_2, ..., a_n)$, and in "after", Y and X have been switched in all voters preference ballots.

Since X was the winner in "before", X got the most number of points (with the possibility that he is tied with others).

In "after", Y gets the same number of votes as X did in "before", X gets the same number of votes as Y did in "before", and all other candidates are unchanged. So, Y must now have the most votes (again, possibly tied with others) in "after". Thus, Y is a winner in "after", showing that the method is neutral.

 <u>Hare's and Coombs' methods are neutral</u>: Suppose we have profiles "before" and "after", and X and Y, as in the definition of neutrality.

If X was the majority winner in the first round of Hare's/Coombs' method in "before", then since X and Y have switched in "after" (and all others have stayed fixed), Y must be the majority winner in the first round for "after".

If there was no majority winner in the first round for "before", then a candidate was eliminated, and that candidate was not X. Thus, Y cannot be eliminated after the first round of "after".

And so on, until either X is the majority winner in a round in "before", in which case Y is in "after", or until all of the remaining candidates have an exact tie, which in "before" must include X, so in "after" it includes Y. Thus, Y will be a winner.

<u>Copeland's method is neutral:</u>

Suppose we have profiles "before" and "after", and X and Y, as in the definition of neutrality.

Note that the results of head-to-head matchups featuring candidates other than X and Y are unchanged in going from "before" to "after".

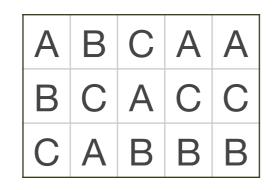
In "after", the scores in each of X's head-to-head matchups become the same as Y's from "before", i.e., if Y beat A 3-2 before, X now beats A 3-2. Likewise, the scores in each of Y's head-to-head mathcups become the same as X's from before.

Thus, Y's tally (1 point for a win, 1/2 point for a tie) in "after" is the same as X's was in "before", and vice-versa. The other candidates tallies will not change. So, if X was a winner "before", Y must be a winner "after".

Monotonicity

- A social choice function is **monotone** if the following holds:
 - Suppose we are given two profiles, "before" and "after".
 - In "before", candidate X is a winner, but a certain voter v places a different candidate Y *immediately above* X on their preference ballot.
 - The "after" profile is identical to "before", except that v now places X *immediately above* Y.
 - Then, X must remain a winner in "after".
- Note: This differs from the two-candidate definition, because it does not require X to be a unique winner.
- The intuition remains the same, however; being preferred *more* by the electorate should not hurt a candidate's chances of winning.

 For example: If our social choice function is monotone, and selects A as a winner in the first profile, it must select A as a winner in the second profile.



A	В	С	Α	Α
В	Α	Α	С	С
С	С	В	В	В

• Likewise in the following two profiles:

A	В	С	Α
В	С	В	С
С	Α	Α	В

Α	В	Α	Α
В	С	С	С
С	А	В	В

 Voter 3 has moved A up two spots, but you can imagine an intermediate profile, and monotonicity says all 3 profiles must select A as a winner.

<u>Proposition:</u> Plurality, Borda count, Copeland and all positional methods are monotone.

• Why?

<u>Proposition:</u> Hare's and Coombs' methods are not monotone.

• Counterexamples?

Positional methods are monotone:

Suppose we are using the positional method $P(a_1, a_2,...,a_n)$. Recall that $a_1 \ge a_2 \ge ... \ge a_n$. Thus, if one voter increases a winner X's position by one spot in their preference ballot from "before", X must receive at least as many (possibly more) points in "after", the demoted candidate Y must receive no more points (possibly fewer), while all of the other candidates point totals are unchanged. Thus, X remains a winner in "after".

<u>Copeland's method is monotone:</u>

By increasing X's position, this can only make X win at least as many (possibly more) head-to-head matchups, while not helping Y or any other candidates. Thus, this can increase (or leave unchanged) X's tally, but not help any other candidates, so X must remain a winner.

• To see that Hare's method is not monotone, consider the following counterexample:

before					af	ter	
6	5	4	2	6	5	4	
А	С	В	В	А	С	В	
В	А	С	А	В	А	С	
С	В	А	С	С	В	А	

- In "after", the last two voters have moved A above B.
- You can check (try it) that C is eliminated first in "before", and then A wins, but in "after", B is eliminated, and then C wins.

Majority

 A social choice function satisfies the majority criterion if whenever a candidate receives a majority of the first-place votes, that candidate must be the unique winner.

<u>Proposition:</u> Plurality, Hare, Coombs* and Copeland satisfy the majority criterion.

- Why? (For Plurality, Hare and Coombs, this is easy. For Copeland, note that a candidate receiving a majority of first-place votes must win in *every* head-to-head matchup.)
- ***Caution**: The version of Coombs in R&U does not, but ours does.

<u>Proposition:</u> Borda count (and thus positional methods, in general) does not satisfy the majority criterion.

Counterexample?

Majority (cont'd)

• To see that the Borda count method does not satisfy the majority criterion, consider the following counterexample:

3	2
Α	В
В	С
С	А

• A receives a majority of first-place votes, but loses to B (7-6) in the Borda count.

Decisiveness

• A social choice function is **decisive** if it always selects a unique winner.

<u>Proposition:</u> Any social choice function that satisfies anonymity and neutrality must violate decisiveness.

• Why? (We'll revisit this, or see p. 54-5 in R&U)

<u>Corollary:</u> Plurality, Borda count (in fact, all positional methods), Hare, Coombs and Copeland all fail to be decisive.

- Recommended reading: Sections 3.1-3.2 in R&U
- <u>Problem set #2</u> has been posted on the course website.