MATH 1340 — Mathematics & Politics

Lecture 5 — June 26, 2015

An example

- (Exercise 2.1 in R&U) Consider the following profile involving 5 candidates and 11 voters:

4	3	2	1	1
A	B	C	D	E
C	E	B	B	D
D	D	D	E	B
B	C	A	C	A
E	A	E	A	C

(a) Who wins the election if the plurality method is used?

(b) Who wins the election if the Borda count method is used?

(c) Who wins the election if the Coombs' method is used?

An example (cont'd)

(a) A wins with a plurality of 4 first-place votes.

(b) With Borda count:
A gets 4x4+0+2x1+0+1x1 = 19 points
B gets 4x1+3x4+2x3+1x3+1x2 = 27 points
C gets 4x3+3x1+2x4+1x1+0 = 24 points
D gets 4x2+3x2+2x2+1x4+1x3 = 25 points
E gets 0+3x3+0+1x2+1x4 = 15 points, so B wins.

(c) With Coombs' method: There is no majority first-place winner, so we eliminate E (having the most last-place votes), and get a new profile.

There is still no majority winner, so we eliminate A, and get a new profile

C is now the majority winner, thus C wins.

4	3	2	1	1
A	B	C	D	E
C	E	B	B	D
D	D	D	E	B
B	C	A	C	A
E	A	E	A	C

4	3	2	1	1
A	B	C	D	D
C	D	B	B	B
D	C	D	C	A
B	A	A	A	C

4	3	2	2
C	B	C	D
D	D	B	B
B	C	D	C

Positional methods

- The following methods generalize the Borda count:

- For n candidates, the **positional method** associated to the sequence of numbers $a_1 \geq a_2 \geq \ldots \geq a_n$ is the social choice function described as follows:
  - Assign, for each candidate, a_1 points to every first-place vote, a_2 points to every second-place vote, and so on, until we assign a_n points for every last-place vote.
  - Tally the points, and the winners are the candidates who receive the most points.

- We denote this method by $P(a_1, a_2, \ldots, a_n)$.

Positional methods (cont'd)

- Observe that the Borda count, for n candidates, is exactly the positional method $P(n\text{-}1, n\text{-}2, \ldots, 1, 0)$.
- The plurality method can be thought of as the positional method $P(1,0,0,\ldots,0)$.
- The **antiplurality method**, which picks as winners those with the *fewest last-place votes*, can be thought of as the positional method $P(1,1,\ldots,1,0)$.

Functional equivalence

- We say that two social choice functions are **functionally equivalent** if whenever they are given the same input profiles, they produce the same result.
- Again, this notion is **not** explicit in the text, but will be used repeatedly.

Criteria

- We again need to specify **criteria** (properties which may or may not hold of a given social choice function) in order to determine which methods are acceptable, and which are not.
- Some of these will be familiar from the two-candidate case, others will be quite new.

Anonymity

- A social choice function is **anonymous** if the following holds:
  - Suppose we are given two profiles, “before” and “after”,
  - the “after” the result of some voters in the “before” *exchanging* their preference ballots.
  - Then, the outcome of the social choice function on these profiles must be the same.

- As in the two-candidate case, anonymity means that *rearranging the ballots does not change the outcome*, so *all voters are treated equally*.

Anonymity (cont'd)

- As in the two-candidate case we have (with a similar proof):

<u>Proposition:</u> A social choice function is anonymous if and only if it depends only on the tabulated profile.

- You can try to write out a detailed proof yourself.

<u>Corollary:</u> Plurality, Borda count, Hare, Coombs, Copeland, and all positional methods are anonymous.

- The dictatorship method is, of course, not anonymous.

Neutrality

- A social choice function is **neutral** if the following holds:
  - Suppose that we are given two profiles, “before” and “after”. In “before”, candidate X is a winner, and
  - for some *other* candidate Y, *all* of the voters have interchanged X with Y in their preference orders in “after”.
  - Then, the function must select Y as a winner in “after”.

- Again, this means that *candidates are treated equally*; if support for one candidate can allow that candidate to win, then the same support can allow any other candidate to win.

Neutrality (cont’d)

Proposition: *Plurality, Borda count, Hare, Coombs, Copeland, and all positional methods all neutral.*

- Why? (See next slides)
- Monarchy is, of course, not neutral.

Neutrality (cont'd)

- Positional methods (thus plurality and Borda count, as well) are neutral:

  Suppose we are given profiles "before" and "after" in which X is a winner with the positional method $P(a_1, a_2,\ldots,a_n)$, and in "after", Y and X have been switched in all voters preference ballots.

  Since X was the winner in "before", X got the most number of points (with the possibility that he is tied with others).

  In "after", Y gets the same number of votes as X did in "before", X gets the same number of votes as Y did in "before", and all other candidates are unchanged. So, Y must now have the most votes (again, possibly tied with others) in "after". Thus, Y is a winner in "after", showing that the method is neutral.

Neutrality (cont'd)

- Hare's and Coombs' methods are neutral:
  Suppose we have profiles "before" and "after", and X and Y, as in the definition of neutrality.

  If X was the majority winner in the first round of Hare's/Coombs' method in "before", then since X and Y have switched in "after" (and all others have stayed fixed), Y must be the majority winner in the first round for "after".

  If there was no majority winner in the first round for "before", then a candidate was eliminated, and that candidate was not X. Thus, Y cannot be eliminated after the first round of "after".

  And so on, until either X is the majority winner in a round in "before", in which case Y is in "after", or until all of the remaining candidates have an exact tie, which in "before" must include X, so in "after" it includes Y. Thus, Y will be a winner.

Neutrality (cont'd)

- Copeland's method is neutral:
  Suppose we have profiles "before" and "after", and X and Y, as in the definition of neutrality.

  Note that the results of head-to-head matchups featuring candidates other than X and Y are unchanged in going from "before" to "after".

  In "after", the scores in each of X's head-to-head matchups become the same as Y's from "before", i.e., if Y beat A 3-2 before, X now beats A 3-2. Likewise, the scores in each of Y's head-to-head mathcups become the same as X's from before.

  Thus, Y's tally (1 point for a win, 1/2 point for a tie) in "after" is the same as X's was in "before", and vice-versa. The other candidates tallies will not change. So, if X was a winner "before", Y must be a winner "after".

Monotonicity

- A social choice function is **monotone** if the following holds:
  - Suppose we are given two profiles, “before” and “after”.
  - In “before”, candidate X is a winner, but a certain voter v places a different candidate Y *immediately above* X on their preference ballot.
  - The “after” profile is identical to “before”, except that v now places X *immediately above* Y.
  - Then, X must remain a winner in “after”.
- **Note:** This differs from the two-candidate definition, because it does not require X to be a unique winner.
- The intuition remains the same, however; being preferred *more* by the electorate should not hurt a candidate’s chances of winning.

Monotonicity (cont'd)

- For example: If our social choice function is monotone, and selects A as a winner in the first profile, it must select A as a winner in the second profile.

A	B	C	A	A
B	C	A	C	C
C	A	B	B	B

A	B	C	A	A
B	A	A	C	C
C	C	B	B	B

- Likewise in the following two profiles:

A	B	C	A
B	C	B	C
C	A	A	B

A	B	A	A
B	C	C	C
C	A	B	B

  - Voter 3 has moved A up two spots, but you can imagine an intermediate profile, and monotonicity says all 3 profiles must select A as a winner.

Monotonicity (cont'd)

Proposition: *Plurality, Borda count, Copeland and all positional methods are monotone.*

- Why?

Proposition: *Hare's and Coombs' methods are not monotone.*

- Counterexamples?

Monotonicity (cont'd)

- Positional methods are monotone:
  Suppose we are using the positional method $P(a_1, a_2, \ldots, a_n)$. Recall that $a_1 \geq a_2 \geq \ldots \geq a_n$. Thus, if one voter increases a winner X's position by one spot in their preference ballot from "before", X must receive at least as many (possibly more) points in "after", the demoted candidate Y must receive no more points (possibly fewer), while all of the other candidates point totals are unchanged. Thus, X remains a winner in "after".

- Copeland's method is monotone:
  By increasing X's position, this can only make X win at least as many (possibly more) head-to-head matchups, while not helping Y or any other candidates. Thus, this can increase (or leave unchanged) X's tally, but not help any other candidates, so X must remain a winner.

Monotonicity (cont'd)

- To see that Hare's method is not monotone, consider the following counterexample:

before

6	5	4	2
A	C	B	B
B	A	C	A
C	B	A	C

after

6	5	4	2
A	C	B	A
B	A	C	B
C	B	A	C

- In "after", the last two voters have moved A above B.
- You can check (try it) that C is eliminated first in "before", and then A wins, but in "after", B is eliminated, and then C wins.

Majority

- A social choice function satisfies the **majority criterion** if whenever a candidate receives a majority of the first-place votes, that candidate must be the unique winner.

Proposition: *Plurality, Hare, Coombs* and Copeland satisfy the majority criterion.*

- Why? (For Plurality, Hare and Coombs, this is easy. For Copeland, note that a candidate receiving a majority of first-place votes must win in *every* head-to-head matchup.)

- ***Caution**: The version of Coombs in R&U does not, but ours does.

Proposition: *Borda count (and thus positional methods, in general) does not satisfy the majority criterion.*

- Counterexample?

Majority (cont'd)

- To see that the Borda count method does not satisfy the majority criterion, consider the following counterexample:

3	2
A	B
B	C
C	A

- A receives a majority of first-place votes, but loses to B (7-6) in the Borda count.

Decisiveness

- A social choice function is **decisive** if it always selects a unique winner.

Proposition: *Any social choice function that satisfies anonymity and neutrality must violate decisiveness.*

- Why? (We'll revisit this, or see p. 54-5 in R&U)

Corollary: *Plurality, Borda count (in fact, all positional methods), Hare, Coombs and Copeland all fail to be decisive.*

- Recommended reading: Sections 3.1-3.2 in R&U
- Problem set #2 has been posted on the course website.