MATH 1340 — Mathematics & Politics

Lecture 6 — June 29, 2015

“Basic” criteria

- A social choice function is **anonymous** if voters exchanging preference ballots doesn’t change winners.
- A social choice function is **neutral** if switching a winner with another candidate in *all* preference orders turns the latter into a winner.
- A social choice function is **monotone** if moving a winner up one position in a voter’s preference order does *not* cause them to become a loser.
- A social choice function satisfies the **majority criterion** if whenever a candidate receives a majority of first-place votes, that candidate is the *unique* winner.
- A social choice function is **decisive** if it only selects *unique* winners.

“Basic” criteria (cont’d)

<u>Proposition:</u> Any social choice function that satisfies anonymity and neutrality must violate decisiveness.

<u>Proof (from class):</u> Suppose that our social choice function is anonymous and neutral, and we have just 2 candidates, and 2 voters (any even number will work). Consider the following profile:

A	B
B	A

If our method was decisive, we must have a unique winner, say A. But now, consider the profile in which we switch candidates A and B:

B	A
A	B

By neutrality, B must be a winner in this second profile. However, this “switch” had the same effect as the voters exchanging their preference ballots. Thus, anonymity implies that A must still be the winner in this profile. But then there is a tie in this second profile, contradicting that our method was decisive. Thus, it must not be decisive.

"Basic" criteria (cont'd)

Proposition: Any social choice function that satisfies anonymity and neutrality must violate decisiveness.

Proof (cont'd): A similar trick works for any number of candidates. Consider the case where we have 3 candidates, and 6 voters. Consider the profile:

A	A	B	B	C	C
B	C	A	C	A	B
C	B	C	A	B	A

Suppose that A was the unique winner. Consider the profile we get from switching A and C:

C	C	B	B	A	A
B	A	C	A	C	B
A	B	A	C	B	C

By neutrality, C must become a winner, but this switch had the same effect as voters exchanging ballots (1 with 6, 2 with 5, 3 with 4), so anonymity implies A is still a winner. Similarly if B or C was the unique winner in the original profile. Thus the method cannot be decisive.

QED

“Basic” criteria (cont’d)

- Here’s where we stand:

	Anonymous	Neutral	Monotone	Majority	Decisive
Plurality	Yes	Yes	Yes	Yes	No
Anti-Plur.	Yes	Yes	Yes	No	No
Borda	Yes	Yes	Yes	No	No
Hare	Yes	Yes	No	Yes	No
Coombs	Yes	Yes	No	Yes	No
Copeland	Yes	Yes	Yes	Yes	No
Dictatorship	No	Yes	Yes	Yes	Yes
Monarchy	Yes	No	Yes	No	Yes
All-ties	Yes	Yes	Yes	No	No

The Pareto criterion

- A social choice function satisfies the **Pareto criterion** if whenever *every* voter prefers one candidate over another, say X over Y, the function does *not* select Y as a winner.

- A candidate is **Pareto optimal** if no *single* other candidate is preferred by *all* voters.
The Pareto criterion demands that *only* Pareto optimal candidates are *permitted* to win.

- This term comes from economics: An alternative is **Pareto optimal** if no *single* other alternative is unanimously preferred.

Vilfredo Pareto
(1848-1923)

The Pareto criterion (cont'd)

- For example: Consider the following profile:

A	B	A	B
C	A	B	A
B	C	C	C

- Since *every* voter prefers A over C, C cannot win if this method is Pareto.
- Both A and B are Pareto optimal; no *single* candidate is preferred by *all* voters over A, and likewise for B.

The Pareto criterion (cont'd)

- However, consider the following profile:

A	B	A	B
C	C	B	A
B	A	C	C

- C is now Pareto optimal, because neither A nor B are *unanimously* preferred over C.

- Being Pareto does not rule out any candidates as winners in this profile.

- **Caution:** The Pareto criterion does **not** say that if a candidate is Pareto optimal then they must win (we'll see that plurality is Pareto, so the above profile would be a counterexample; C is Pareto optimal but is not a winner in the plurality method), only that *non-Pareto candidates must not win*.

The Pareto criterion (cont'd)

- A social choice function satisfies the **unanimity criterion** if whenever a candidate receives *every* first-place vote, they are selected as the unique winner.

- If a social choice function satisfies the majority criterion, it must satisfy the unanimity criterion. Why?

  - From class: If a candidate receives *every* first-place vote, then they've received a majority of first-place votes.

- (Exercise 3.5 in R&U) If a social choice function satisfies the Pareto criterion, it must satisfy the unanimity criterion. Why?

The Pareto criterion (cont'd)

- (Exercise 3.5 in R&U) If a social choice function satisfies the Pareto criterion, it must satisfy the unanimity criterion. Why?

- From class: Suppose that candidate X gets every first-place vote. Can any other candidate Y win?
  Nope! If Y is any other candidate, then every voter places X above Y (since Y gets all first-place votes). Thus Y is not Pareto optimal, so Y cannot win if our method is Pareto. Since someone must win (social choice functions cannot be indifferent), it must be X.

The Pareto criterion (cont'd)

Proposition: *The plurality method satisfies the Pareto criterion.*

- Why?
- Is dictatorship Pareto? Monarchy?

The Pareto criterion (cont'd)

<u>*Proposition:*</u> *The plurality method satisfies the Pareto criterion.*

<u>Proof (from class):</u> Suppose that we are given a profile in which *every* voter ranks candidate X over candidate Y. We must show that Y *cannot* win in the plurality method.

How does the plurality method pick winners? By finding the candidate with the most first-place votes.

But in this profile, candidate Y gets *no* first-place votes, because they are always ranked behind X! Thus, candidate Y cannot win in the plurality method.

QED

The Pareto criterion (cont'd)

Dictatorships are Pareto: If candidate X is ranked above candidate Y in every profile, then in particular, this is true in the dictator's profile, so Y cannot be the dictator's first choice, and thus does not win.

Monarchies are not Pareto: Consider the following counterexample: Suppose A is the monarch, but we are given the profile

B	B	C
C	C	B
A	A	A

A is the winner, but is clearly not Pareto optimal.

The Pareto criterion (cont’d)

Proposition: *The anti-plurality method does not satisfy the Pareto criterion.*

Consider the following counterexample (from class):

A	A	A
B	B	B
C	C	C

In the anti-plurality method, A and B are both winners, but B is not Pareto optimal (since A is unanimously preferred).

The Condorcet candidate

- Given a profile, a candidate is called a **Condorcet candidate** if he or she beats every other candidate in a head-to-head (simple majority) match-up.
- Note, not every profile has a Condorcet candidate, for example:

A	C	B
B	A	C
C	B	A

Nicolas de Condorcet
(1743-1794)

- Consequently, the “method”, “select the Condorcet candidate as the winner”, is not actually a social choice function.
- However, if there is a Condorcet candidate, they are unique. Why?
  - If there were two Condorcet candidates, each would defeat the other in a head-to-head match-up, which is impossible.

Condorcet criteria

- A social choice function satisfies the **Condorcet criterion** if whenever a profile *has* a Condorcet candidate, the methods selects that candidate as the unique winner.
- Similar in spirit: A candidate is the **anti-Condorcet candidate** if he or she *loses* to every other candidate in a head-to-head match-up.
  - The same example shows that not every profile has an anti-Condorcet candidate.
- A social choice function satisfies the **anti-Condorcet criterion** if whenever a profile *has* an anti-Condorcet candidate, the methods must *not* select that candidate as a winner.

Condorcet criteria (cont'd)

- (Exercise 3.1 in R&U) Consider the profile

A	D	E	C	D
B	B	B	A	E
C	E	A	B	C
D	A	C	E	B
E	C	D	D	A

- We claim that B is the Condorcet candidate. Why? Consider the head-to-head match-ups:

3	2
B	A
A	B

3	2
B	C
C	B

3	2
B	D
D	B

3	2
B	E
E	B

- Since B wins each of these match-ups, B is the Condorcet candidate

Condorcet criteria (cont'd)

- (Exercise 3.3 in R&U) Can you come up with a profile that has an anti-Condorcet candidate, but no Condorcet candidate?

- Consider the example profile (from class):

A	B	A	B
B	A	B	A
C	C	C	C

- C is anti-Condorcet (losing to A and B 4-0), but neither A nor B are Condorcet (since they tie 2-2).

Condorcet criteria (cont'd)

Proposition: *The plurality method does not satisfy the Condorcet nor the anti-Condorcet criteria.*

- Counterexamples?
- Dictatorship? Monarchy?

Condorcet criteria (cont'd)

Proposition: The plurality method does not satisfy the Condorcet nor the anti-Condorcet criteria.

Proof (differs from class): Consider the following tabulated profile:

3	2	2
A	B	C
B	C	B
C	A	A

A is the unique plurality winner. However, B is the Condorcet candidate since B beats A 4-3 and beats C 5-2. Thus, this method fails the Condorcet criterion.

In fact, A is anti-Condorcet in this example! She loses 3-4 to C as well. Thus, this example also shows that this method fails the anti-Condorcet criterion.

QED

Condorcet criteria (cont'd)

<u>Dictatorships are not Condorcet or anti-Condorcet:</u> Use the profile on the previous slide, with A as the first voter as the dictator. Then A wins, despite B being the Condorcet candidate, and A being anti-Condorcet.

<u>Monarchies are not Condorcet or anti-Condorcet:</u> Use the same example with A the monarch.

Condorcet criteria (cont'd)

Proposition: Copeland's method satisfies both the Condorcet and anti-Condorcet criteria.

Proof (from class): We'll do Condorcet first. Suppose we are given a profile with a Condorcet candidate X. We must show that X is the unique winner with Copeland's method.

Since X beats every opponent in a head-to-head match-up, X gets the maximum possible number of points in Copeland's method (which is n-1, if there are n candidates, one for each match X wins). (R&U calls this a "perfect score".) Meanwhile, every other candidate gets at least one *fewer* points, that is, *no more than* n-2 points, since they lose at least one match-up (namely, the one against X). Thus, X must be the unique winner.

To see that Copeland satisfies anti-Condorcet, note that an anti-Condorcet candidate Y gets 0 points, since they lose every match-up, and the other candidates get at least 1 point (from the match against Y), so Y cannot win.

QED

- Recommended reading: Section 3.3 in R&U
- Problem set #2 is due tomorrow in class.
- Solutions to Problem set #1 have been posted on the course website.