MATH 1340 — Mathematics & Politics

Lecture 7 — June 30, 2015

Condorcet criteria

- Recall from last time: A social choice function satisfies the **Condorcet criterion** if whenever there is a **Condorcet candidate** (a candidate who wins all head-to-head match-ups), that candidate must be the *unique* winner.

- We have already seen that the Pareto criterion implies the unanimity criterion; something similar happens with Condorcet.

<u>*Proposition:*</u> *If a social choice function satisfies the Condorcet criterion, then it satisfies the majority criterion.*

- Why?

Condorcet criteria (cont'd)

<u>Proposition:</u> If a social choice function satisfies the Condorcet criterion, then it satisfies the majority criterion.

Proof (from class): Suppose that our method satisfies the Condorcet criteria, and we are given a profile in which candidate A gets a majority of first-place votes. We need to show that A is the unique winner.

Since out method is Condorcet, it suffices to show that A is the Condorcet candidate.

Since A gets a majority of first-place votes, in a head-to-head match-up with any other candidate, A must get still get a majority, because in each of those ballots that she is placed first, she beats the other candidate. (She may get more in the head-to-head match-up.) Thus, A beats any other candidate, so is Condorcet, and thus the unique winner.

QED

Condorcet criteria (cont'd)

- (Exercise 4.2 in R&U) Consider the social choice function called the COP method (Condorcet o/w Plurality): If a profile has a Condorcet candidate, select them as the winner. Otherwise, select the plurality winner(s).

(a) Does this method satisfy the Condorcet criterion?

(b) Does this method satisfy the anti-Condorcet criterion?

From class:

(a) Yes, this method selects the Condorcet candidate, if there is one, as the unique winner, which means it satisfies the Condorcet criterion by definition.

(b) No, consider the following counterexample:
A is anti-Condorcet (beaten by B, C and D 3-2),
no candidate is Condorcet, but
A is a plurality winner (with C).

A	A	B	C	C
D	D	D	B	B
C	B	C	D	D
B	C	A	A	A

Independence

- Consider the following profile:

12%	37%	25%	22%	4%
A	A	C	C	B
B	C	A	B	C
C	B	B	A	A

- Using the plurality method, A is the unique winner with 49% of first-place votes.

- However, if the voters who ranked candidate B first instead ranked candidate C first, C would win (and A lose), even though *no one* has changed their preference of A over C, or C over A.

- Voters opinions about candidate B can have an effect on the outcome, even though B is an "irrelevant alternative" as far as A and C are concerned (think about the Bush, Gore and Nader example).

Independence (cont'd)

- In 1950, Kenneth Arrow established a criterion for avoiding this situation. He would go on to win the 1972 Nobel Prize in Economics, in part for his work on social choice theory.

Kenneth J. Arrow
(1921-)

- A social choice function satisfies the **independence criterion** (or **independence of irrelevant alternatives**) if the following holds:
  - Suppose we are given two profiles, "before" and "after",
  - in which there are two candidates, X and Y, such that no voter changes their preference of X over Y, and vice-versa (i.e., if a voter places X over Y in "before", they must put X over Y in "after", and vice-versa),
  - and in "before" candidate X is a winner, but candidate Y is not.
  - Then, Y must *not* be a winner in "after".

Independence (cont'd)

- A social choice function satisfies the **independence criterion** if the following holds:
  - Suppose we are given two profiles, "before" and "after",
  - in which there are two candidates, X and Y, such that no voter changes their preference of X over Y, and vice-versa (i.e., if a voter places X over Y in "before", they must put X over Y in "after", and vice-versa),
  - and in "before" candidate X is a winner, but candidate Y is not.
  - Then, Y must *not* be a winner in "after".
- The idea: If X defeats Y, then voters changing their minds about the *other candidates*, but not their *relative preferences* between X and Y, should not allow Y to become a winner.

Independence (cont'd)

- The example on the previous slide (also, the 2000 Presidential Election in Florida) shows:

Proposition: *The plurality method does not satisfy independence.*

- The Condorcet criterion is also at odds with independence:

Proposition: *No social choice function involving at least 3 candidates satisfies both the independence criterion and Condorcet criterion.*

- Why?

Independence (cont'd)

Proposition: *No social choice function involving at least 3 candidates satisfies both the independence criterion and Condorcet criterion.*

Proof (from class, for 3 candidates): Suppose we have a social choice function (for 3 candidates) which satisfies independence and Condorcet. We will show that this leads to a contradiction, thus cannot happen.

Consider the following "before" and "after" pair:

before

A	C	C
B	A	B
C	B	A

after

A	C	B
B	A	C
C	B	A

C is the Condorcet candidate in "before", thus wins (in particular, defeats A.) Since C and A have the same relative positions in "after", independence dictates that A cannot be a winner in "after".

Independence (cont'd)

<u>*Proposition:*</u> *No social choice function involving at least 3 candidates satisfies both the independence criterion and Condorcet criterion.*

<u>Proof (cont'd):</u> Consider a second "before" and "after" pair, with the same "after" as before:

A is Condorcet in "before", thus wins and defeats B, so independence dictates that B cannot win in "after".

before

A	A	C
B	C	B
C	B	A

after

A	C	B
B	A	C
C	B	A

Lastly, consider:

B is Condorcet in "before", thus wins and defeats C, so independence dictates that C cannot win in "after".

before

B	C	B
A	A	C
C	B	A

after

A	C	B
B	A	C
C	B	A

But now, we have ruled out all possible candidates in the "after" profile (which has been the same in each case). This is a contradiction. QED

Independence (cont'd)

- Since Copeland's method satisfies the Condorcet criterion, we have:

<u>Corollary:</u> *Copeland's method does not satisfy independence.*

- Do any of our methods satisfy independence?

<u>Proposition:</u> *The dictatorship, monarchy and all-ties methods satisfy independence.*

- Why? (Think about this, or look in section 4.2 of R&U)

Criteria

- Where we stand now:

	Anon.	Neut.	Mon.	Maj.	Dec.	Par.	Cond.	Anti-Cond.	Ind.
Plurality	Yes	Yes	Yes	Yes	No	Yes	No	No	No
Anti-Plur.	Yes	Yes	Yes	No	No	No			
Borda	Yes	Yes	Yes	No	No		?		
Hare	Yes	Yes	No	Yes	No				
Coombs	Yes	Yes	No	Yes	No				
Copeland	Yes	Yes	Yes	Yes	No		Yes	Yes	No
Dictatorship	No	Yes	Yes	Yes	Yes	Yes	No	No	Yes
Monarchy	Yes	No	Yes	No	Yes	No	No	No	Yes
All-ties	Yes	Yes	Yes	No	No	No	No	No	Yes

- Goal: Fill in the blanks.

Anti-plurality

Proposition: *The anti-plurality method does not satisfy:*

(1) the Condorcet criterion,

(2) the anti-Condorcet criterion,

(3) the independence criterion.

Counterexamples:
Anti-plurality is not Condorcet or anti-Condorcet: Consider the profile

C	C	C	B	B
A	A	B	A	A
B	B	A	C	C

C is the Condorcet candidate, but A wins in the anti-plurality method. Moreover, A is anti-Condorcet! So this method is neither Condorcet or anti-Condorcet.

Anti-plurality (cont'd)

Proposition: The anti-plurality method does not satisfy:

(1) the Condorcet criterion,

(2) the anti-Condorcet criterion,

(3) the independence criterion.

Counterexamples:

Anti-plurality is not independent: Consider the profiles:

before

C	C	C	B	B
A	A	B	A	A
B	B	A	C	C

after

A	A	C	B	B
B	B	B	A	A
C	C	A	C	C

A wins in "before", and defeats B. However, B wins in "after", even though the relative positions of A and B have not changed. Thus, this method is not independent.

Borda count

Proposition: *The Borda count method does not satisfy:*

(1) the Condorcet criterion,

(2) the independence criterion.

Proposition: *The Borda count method satisfies:*

(1) the Pareto criterion,

*(2) *the anti-Condorcet criterion.*

*This fact is more difficult than the others in this section; we'll come back to it.

Borda count (cont'd)

Proposition: *The Borda count method does not satisfy:*

(1) the Condorcet criterion,

(2) the independence criterion.

Counterexamples:
Borda count is not Condorcet: Consider the profile:

A	A	A	B	B
B	B	B	C	C
C	C	C	A	A

A is the Condorcet candidate, but only gets 6 Borda points, while B gets 7 points, so B wins. Thus, this method cannot be Condorcet.

Borda count (cont'd)

<u>Proposition:</u> The Borda count method does not satisfy:

(1) the Condorcet criterion,

(2) the independence criterion.

Counterexamples:
<u>Borda count is not Independent:</u> Consider the profiles:

before

A	A	A	B	B
B	B	B	C	C
C	C	C	A	A

after

A	A	A	B	B
B	B	C	C	C
C	C	B	A	A

We've seen that B wins in "before", and defeats A.
However, in "after", A (together with B) wins. Since the relative positions of A and B have not changed, this shows that this method cannot be independent.

- Recommended reading: Finish Section 3.3, then start reading Sections 4.1-4.2 in R&U

- Problem set #3 has been posted on the website and is due on **Thursday**, in class or by 4pm. (You can slide it under my office door, 112 Malott Hall.)