MATH 1340 — Mathematics & Politics

Lecture 8 — July 1, 2015

An example

- (Exercise 3.7 in R&U) Consider the two profiles (same electorate, assume the method is anonymous):

Profile P

7	5	4	1
A	C	B	B
B	A	C	A
C	B	A	C

Profile Q

8	5	4
A	C	B
B	A	C
C	B	A

(a) Suppose that a method choose candidate A as the unique winner in Profile P, but candidate B as the unique winner in Profile Q. What can you say about such a method?

(b) Suppose that a method chooses candidate A as the unique winner in Profile P, but candidate C as the unique winner in Profile Q. What can you say about such a method?

An example (cont'd)

Profile P

7	5	4	1
A	C	B	B
B	A	C	A
C	B	A	C

Profile Q

8	5	4
A	C	B
B	A	C
C	B	A

(a) The method is not monotone: The last voter in Profile P placed candidate B directly above A, but in Profile Q (assuming anonymity) places A directly above B, and the rest is unchanged. If the method was monotone, then A would still be a winner in Profile Q.

(b) The method is not independent: In Profile P, A wins and C loses. In Profile Q, the relative positions of A and C have not changed; there are still 8 voters who place A above C, and 9 voters who place C above A. However, C is now a winner, which violates independence.

Borda count

Proposition: *The Borda count method does not satisfy:*

(1) the Condorcet criterion,

(2) the independence criterion.

Proposition: *The Borda count method satisfies:*

(1) the Pareto criterion,

*(2) *the anti-Condorcet criterion.*

*This fact is more difficult than the others in this section. (See Proposition 4.3 in R&U for the proof.)

Borda count (cont'd)

Borda count is Pareto: Suppose that every voter ranks candidate X above candidate Y. We must show that Y cannot win in the Borda count.

Observe that from each voter, X receives more points than Y (at least one more), so overall, X receives more Borda points than Y. In particular, Y cannot have the most Borda points. Thus, Y cannot be the winner.

QED

Hare’s method

Proposition: *Hare’s method does not satisfy:*

(1) the Condorcet criterion,

(2) the anti-Condorcet criterion,

(3) the independence criterion.

Proposition: *Hare’s method satisfies the Pareto criterion.*

Hare's method (cont'd)

(1) Hare is not Condorcet: Consider the profile:

3	2	2
C	A	B
A	B	A
B	C	C

A is the Condorcet candidate (beating B 5-2, and C 4-3), but in Hare's method, A and B are eliminated in the first round and C wins.

(2) Hare is not anti-Condorcet: Consider the profile:

3	2	2
A	B	C
B	C	B
C	A	A

A is the anti-Condorcet candidate (losing to B 3-4, and C 4-3), but in Hare's method, B and C are eliminated in the first round and A wins.

Note that (coincidentally) these two examples are basically the same.

Hare's method (cont'd)

(3) Hare is not independent: Consider the profiles:

before

2	2	1
B	A	A
A	C	B
C	B	C

after

2	2	1
B	C	A
A	A	B
C	B	C

In "before", A wins (having a majority of first-place votes), and B loses.

In "after", B wins (after A is defeated in the first round).

Observe that the relative positions of A and B are the same in both; the first two voters prefer B to A and the last 3 prefer A to B. Since B goes from being defeated by to A to being a winner, this shows the method cannot be independent.

Hare’s method (cont’d)

Hare is Pareto: Suppose that every voter ranks candidate X over candidate Y. Then candidate Y must get no first-place votes, being ranked below X by everyone.

Thus, either there is a majority winner, which cannot be Y, or Y is eliminated in the first round. In either case, Y cannot be a winner, so the method is Pareto.

Copeland's method

Proposition: Copeland's method does not satisfy the independence criterion.

- We have already seen that this follows the fact that Copeland satisfies the Condorcet criterion, however an explicit counterexample can be informative.

Proposition: Copeland's method satisfies the Pareto criterion.

Copeland's method (cont'd)

Copeland is not independent: Consider the following profiles:

before

B	A	A
C	B	C
A	C	B

after

B	A	C
C	B	A
A	C	B

In "before", A wins (getting 2 Copeland points for beating both B and C), and B loses.

In "after", A, B and C all win since each gets 1 Copeland point.

Observe that the relative positions of A and B are the same in both. Since B goes from being defeated by to A to being a winner (even though they tie, B is still "a" winner, which is good enough), this shows the method cannot be independent.

Copeland's method (cont'd)

Copeland is Pareto: Suppose that every voter ranks candidate X over candidate Y. Then, every opponent that Y beats in a head-to-head match-up, X must also beat (since each time Y is above, say Z, X is as well). Similarly, if Y ties an opponent, X must either tie or defeat them.

Thus, X gets at least the number of Copeland points that Y gets, plus 1, since X also beats Y. In particular, X has more Copeland points than Y, so Y cannot have the most points.

Thus, Y does not win in this method.

QED

Coombs' method

Proposition: *Coombs' method does not satisfy:*

(1) the Condorcet criterion,

(2) the anti-Condorcet criterion,

(3) the independence criterion.

Proposition: *Coombs' method satisfies the Pareto criterion.*

Note: I will post explanations for these facts on the website later on.

Criteria

- Where we stand:

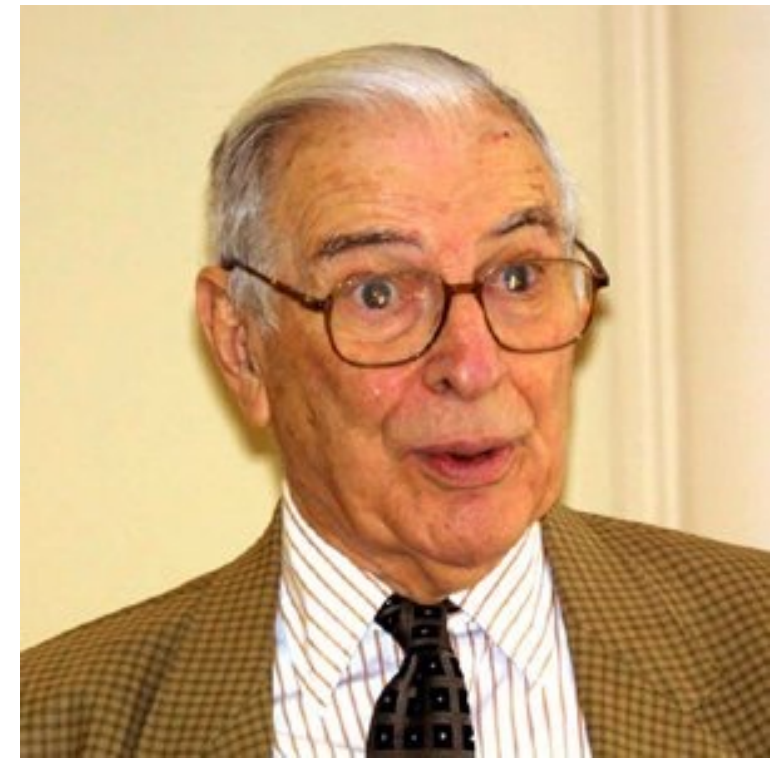
	Anon.	Neut.	Mon.	Maj.	Dec.	Par.	Cond.	Anti-Cond.	Ind.
Plurality	Yes	Yes	Yes	Yes	No	Yes	No	No	No
Anti-Plur.	Yes	Yes	Yes	No	No	No	No	No	No
Borda	Yes	Yes	Yes	No	No	Yes	No	Yes	No
Hare	Yes	Yes	No	Yes	No	Yes	No	No	No
Coombs	Yes	Yes	No	Yes	No	Yes	No	No	No
Copeland	Yes	Yes	Yes	Yes	No	Yes	Yes	Yes	No
Dictatorship	No	Yes	Yes	Yes	Yes	Yes	No	No	Yes
Monarchy	Yes	No	Yes	No	Yes	No	No	No	Yes
All-ties	Yes	Yes	Yes	No	No	No	No	No	Yes

- Do any methods stand out? Do any seem no longer acceptable?

The Big Question

- Are there *any* reasonable voting methods which satisfy the independence criterion?

The Big Answer



Arrow's Impossibility Theorem (1950): *If a social choice function with at least three candidates satisfies Pareto and independence, then it is (functionally equivalent to) a dictatorship.*

Corollary: It is impossible for a social choice function with at least three candidates to satisfy Pareto, independence and anonymity.

- Consequently, one cannot expect reasonable voting methods to satisfy both Pareto and independence.

Arrow’s Impossibility Theorem

Arrow’s Impossibility Theorem (1950): If a social choice function with at least three candidates satisfies Pareto and independence, then it is (functionally equivalent to) a dictatorship.

- This result is the most famous of modern social choice theory. It was contained in Arrow’s doctoral dissertation, and was cited as a reason he was awarded the Nobel Prize.

- Recommended reading: Finish reading Section 4.2, and read Sections 5.1-5.2 in R&U

- Solutions to Problem set #2 will be posted later today.

- Reminder: Problem set #3 is due tomorrow, either in class, to me, or by 4pm underneath my office (112 Malott) door.