MATH 1340 — Mathematics & Politics

Lecture 9 — July 2, 2015

From Problem set #2:

- (Exercise 2.10 in R&U) With 4 candidates, the Borda count is the positional voting method that we denote $P(3,2,1,0)$.

(a) Consider instead the method $P(4,2,1,0)$. Show by example that this not give the same outcome as the Borda count method.

(b) Consider now the method $P(4,3,2,1)$. Explain why this always gives the same outcome as the Borda count.

(c) Consider the method $P(8,6,4,2)$. Explain why this always gives the same outcome as the Borda count.

From Problem set #2: (cont'd)

(a) Consider the profile:
Borda: A and B win, both having 8 points.
P(4,2,1,0): A wins 10 points, B loses having only 9 points.

2	1	1
A	B	C
B	A	D
C	C	B
D	D	A

(b) Each candidate gets an addition point from each voter, so their point total is m+n, where m is their Borda total, and n is the number of voters. Since this happens to every candidate, this doesn't change who has the most points.

(c) Each candidate will get twice the points as they got in (b), but this doesn't change who has the most point as either. So the output is the same as in (b), which was the same as Borda.

An example

- (Exercise 4.2(c) in R&U) Recall the COP method: If there is a Condorcet candidate, then that candidate is the unique winner. Otherwise, select the plurality winner(s). Is this method independent?

Counterexample from class: Consider the profiles

before

B	C	B
A	A	C
C	B	A

after

A	C	B
B	A	C
C	B	A

B wins in “before”, being the Condorcet candidate, defeating C. However, A, B and C all tie for a win in “after” (using plurality). The relative positions of B and C are the same in the two profiles, so this contradicts independence.

Aside: The Condorcet Paradox

- In many ways, the fundamental problem in voting and social choice theory is the following:
- While (we assume) each *individual* holds coherent preferences, it may be impossible to obtain coherent preferences *collectively*.

- The simplest example is the profile:
  - a majority prefer A to B,
  - a majority prefer B to C,
  - and a majority prefer C to A.

A	C	B
B	A	C
C	B	A

- This is the **Condorcet paradox**. Even though voters hold rational (i.e., transitive) preferences, they collectively do not.

- Rational voters may collectively be irrational.

The Condorcet Paradox (cont’d)

- A variation on that example is as follows (from p. 81 in R&U): Suppose there are three closely related bills, A, B and C, being debated in the House of Representatives.

- Everyone agrees that one (and only one) of the bills must pass.

- The representatives hold the following preferences:

145	145	145
A	C	B
B	A	C
C	B	A

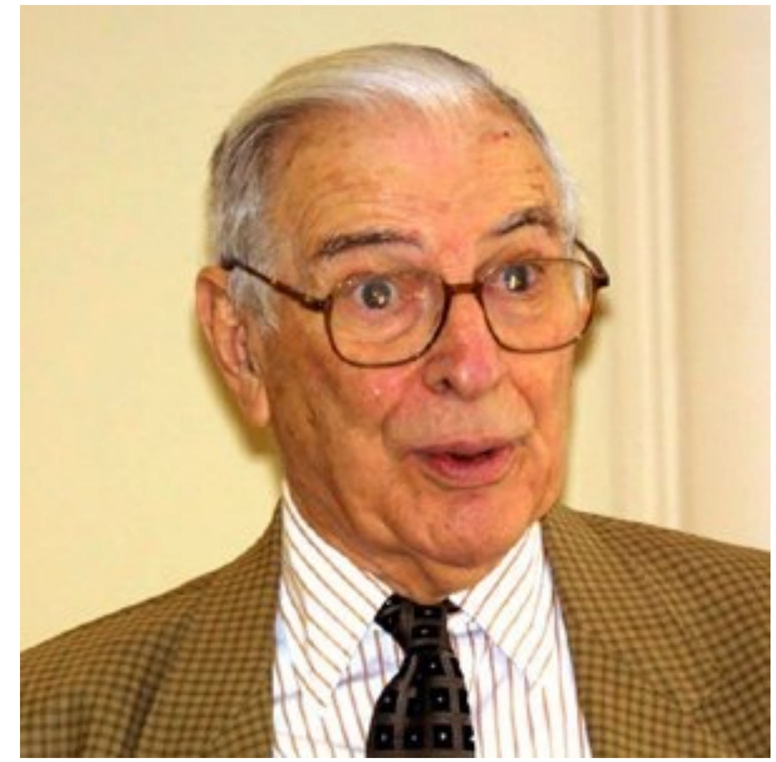
- Debate begins. First bill C is considered, but there is an outcry by an overwhelming (2/3) majority that bill B is superior!

- So, bill B is considered instead. But there is an outcry from an overwhelming majority that bill A is superior!

- So, bill A is considered. But there is an outcry from an overwhelming majority that bill C is superior…

- This difficulty (in a hidden way) contributes to the proof of Arrow’s Theorem.

Arrow’s Theorem



Arrow’s Impossibility Theorem (1950): *If a social choice function with at least three candidates satisfies Pareto and independence, then it is (functionally equivalent to) a dictatorship.*

Corollary: *It is impossible for a social choice function with at least three candidates to satisfy Pareto, independence and anonymity.*

- (Exercise 5.1 in R&U) Note that this Corollary is false if any of the three criteria are removed. Examples?
  - Monarchy is independent and anonymous. Dictatorship is Pareto and independent. Plurality is Pareto and anonymous.

An example

- (Exercise 5.2 in R&U) Consider the method that declares candidates to be winners if and only if they have at least one first-place vote.

(a) Is this method anonymous?
Yes, it only counts votes.

(b) Does this method satisfy the Pareto criterion?
Yes, if candidate X as above Y in all preference ballots, candidate Y gets no first-place votes, and thus loses.

(c) Does this method satisfy the independence criterion?
No! By Arrow's theorem, *no* method is anonymous, Pareto and independent, so certainly this method cannot be. No counterexample is necessary.

Decisiveness

- Our first step towards proving Arrow's theorem will be the following surprising fact:

Lemma (The decisiveness lemma): A social choice function with at least three candidates that satisfies Pareto and independence must be decisive.

- For the proof, see Section 5.3 of R&U or your notes from class.

- Since anonymity and neutrality imply the failure of decisiveness (see Lecture 6), we have:

Corollary: It is impossible for a social choice function with at least three candidates to satisfy anonymity, neutrality, Pareto and independence.

Test information

- Test #1 (of 2) will be on **Wednesday, July 8, in class**.
- If you are not able to attend class (for a *serious* reason) that day, **you must contact me by Monday** at the latest.
- The test will cover the material up to and including Monday's class (chapters 1 through 5 in R&U).
  - If a definition from the lecture differed from the book, use the one from lecture.
- Tuesday will be a review/problem class (there will not be a new problem set).
- You are allowed to have a "cheat sheet" (of definitions, etc).
  - It must be **hand-written and one-sided**, on 8 1/2" x 11" (A4, or similar) paper.
- No calculators, cell phones, textbooks, other notes, etc.

Study advice

- Go through the lectures, know the definitions (criteria!) and important results.
  - Methods to know (Two candidate): simple majority, super majority, status quo, weighted, dictatorship, monarchy, all-ties
  - (Multi-candidate): Plurality, Borda count, Hare, Copeland, positional, dictatorship, monarchy, all-ties.
- Do problems! Try to re-do old homework problems (solutions will be/are posted), and odd-numbered problems from the text (answers are in the back of the book).
- Email me questions on the weekend (some I will do in class on Tuesday) and come to office hours on Monday.

- Recommended reading: Sections 5.2-5.3 in R&U
- Problem set #3 is due today, either in class, to me, or by 4pm underneath my office (112 Malott) door.
- Problem set #4 will be posted on the website later today. It is due Tuesday, in class.