MATH 1340 — Mathematics & Politics

Lecture 13 — July 9, 2015

Apportionment

A survey

“All legislative Powers herein granted shall be vested in a Congress of the United States, which shall consist of a Senate and House of Representatives.”

–Article 1, Section 1, The Constitution of the United States of America

“The House of Representatives shall be composed of Members chosen every second Year by the People of the several States…

Representatives and direct Taxes shall be apportioned among the several States which may be included within this Union, according to their respective Numbers, which shall be determined by adding to the whole Number of free Persons, including those bound to Service for a Term of Years*, and excluding Indians not taxed, three fifths of all other Persons*. The actual Enumeration shall be made within three Years after the first Meeting of the Congress of the United States, and within every subsequent Term of ten Years, in such Manner as they shall by Law direct. The Number of Representatives shall not exceed one for every thirty Thousand, but each State shall have at Least one Representative…”

–Article 1, Section 2, The Constitution of the United States of America

*removed by Section 2 of 14th Amendment, 1868

The Apportionment Problem

- The US Constitution dictates the members (or seats) of the House of Representatives be distributed to the states "according to their respective Numbers", and this carried out every 10 years (upon tallying of the US Census).

- Since 1959 there have been 50 states in US, and while not mandated by the Constitution, the number of House seats has been fixed at 435 since the 1920s.

Problem: How do we distribute these 435 representatives amongst the 50 sates "according to their respective Numbers"?

The Apportionment Problem (cont'd)

- At first, there seems to be an obvious answer:
- For instance: According to the 2010 US Census, there are 308,143,815 people (citizens and non-citizens) in the 50 US states, and 19,378,102 in the state of New York.
- New York has a fraction of 19,378,102/308,143,815, or approximately 6.289%, of the US population, so they *should* get 6.289% of the 435 representatives, which is approximately 27.4.
- But 27.4 is not a *whole number*, and we can't "split" House seats.
- If we round down to 27, New Yorkers will be slightly underrepresented in Congress. If we round up to 28, they will be slightly overrepresented.

The Apportionment Problem (cont'd)

- It gets worse. If we try and round the "ideal" amount of representatives for each state in the usual way, the total number of representatives will not add up to 435 (it adds up to 431 with 2010 data).

- If we try a "generous approach" and round every state up, we get 462 representatives.

- If we decided that 462 is a reasonable total, we would have to start over again with new "ideal" amounts, and have the possibility of an *ever increasing* total number.

The Apportionment Problem (cont'd)

- Ideally, each congressional district should have a population of 308,143,815/435, or about 708,377.
- For some states, the stakes of congressional representation are very high:
- Montana has a population of 989,415, or 0.321% of the total US population.
- Montana's ideal representation is 0.321% of 435, or about 1.4, but with only 1 representative, they'd have *much less* representation in Congress than the ideal district, and with 2 representatives, they'd have *much more*.

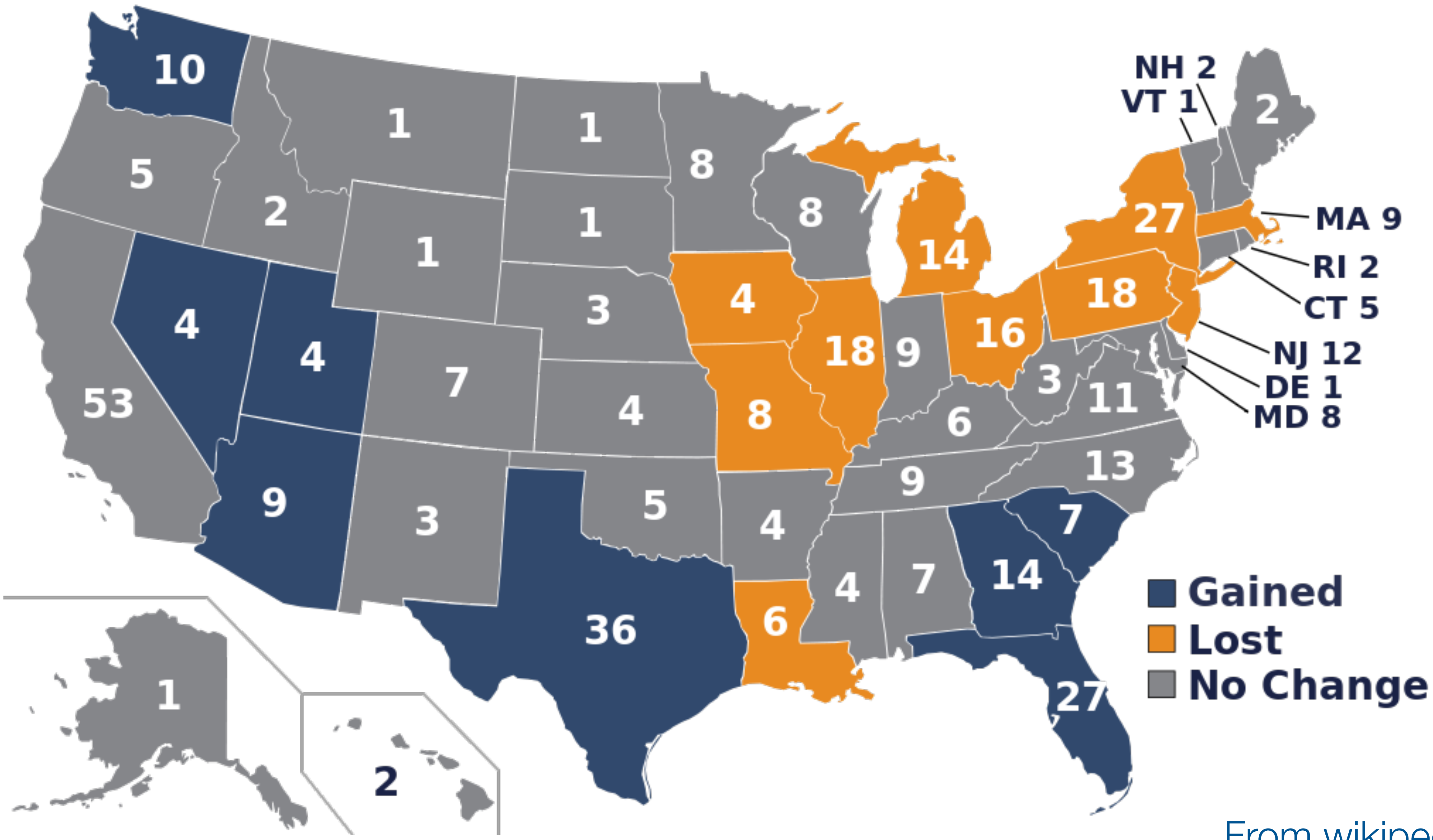
The Electoral College

- This apportionment also affects the method by which the US elects its President:

- Each state is assigned **electors** in the Electoral College equal to the total number of representatives and senators they have in Congress (i.e., number of House seats + 2), with DC getting the same as the smallest state (3).

- (Roughly) The plurality winner of the Presidential election in each state determines who the electors of the state will vote for (as a block), and the plurality winner of the electors' votes becomes the President.

Current (2010 Census) apportionment



From wikipedia

Apportionment problems

- In general, an **apportionment problem** occurs when we wish to distribute a collection of *identical*, *discrete* objects (e.g., House seats) to various stakeholders (e.g., states) with differing claims to those objects (e.g., due to their populations).
  - We will focus on apportioning US House seats, but there are many other applications.
- More precisely, associated to such a problem we have:
  - some whole number $n>1$ of states, numbered 1 through n,
  - state 1 has population p_1, state 2 has population p_2, and so on, so that the total population is $p = p_1+p_2+\ldots+p_n$,
  - and some whole number $h>0$ of the discrete objects we wish to distribute.

Apportionment methods

- Much like with voting, apportionment methods are kinds of functions:
- An **apportionment method** is a function which takes as input the values h, n, $p_1, p_2, \ldots, p_n$, where h and n are positive integers, p_k's are positive numbers, and whose output is a sequence of *nonnegative integers* $a_1, a_2, \ldots, a_n$ such that $h = a_1 + a_2 + \ldots + a_n$.
- The interpretation is that given h objects, and n states with populations $p_1, p_2, \ldots, p_n$, the method distributes a_k objects to the k^{th} state.

Apportionment methods (cont'd)

- For congressional apportionment: $h = 435$ (number of seats in the House), $n = 50$ (number of states), and we may order the states alphabetically so that p_k is the population of the k^{th} state in alphabetical order.

- The output of an apportionment method will consist of how many seats each state is to be given.

- For instance: In alphabetical order, New York is the 32nd state, so $p_{32} = 19{,}378{,}102$.

Quotas

- The quantity $q_k = h(p_k/p)$ is the **standard quota** for the k^{th} state. It is the *ideal* amount to be distributed to this state if we could divide the objects arbitrarily.
- In the case of New York, we have seen that this quantity is $q_{32} = 435(19{,}378{,}102/308{,}143{,}815) \approx 27.4$.
- Since a_k must be a whole number, there are two natural candidates given by rounding down q_k, called the **lower quota**, and rounding up q_k, called the **upper quota**.
- For New York, the lower quota is 27, the upper quota is 28.

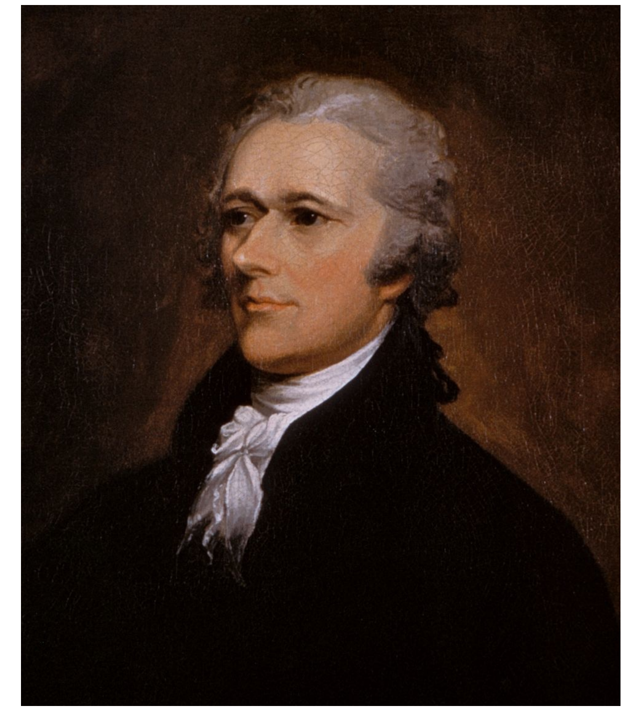
Standard divisor

- We can re-write $q_k = h(p_k/p) = p_k/(p/h)$.
- The quantity $s = p/h$ is called the **standard divisor**, it is the ideal amount of the population entitled to each object.
- In congressional apportiontment, this quantity represents the size of an ideal congressional district. As of 2010, $s = 308{,}143{,}815/435 \approx 708{,}377.$

Hamilton's method

- The Constitution explicitly specified the initial apportionment to the States, but it became an issue for Congress in 1791.

- Alexander Hamilton (founding father and Secretary of the Treasury) proposed what we now call **Hamilton's method**:
  - First, assign to each state its lower quota of seats (there will likely be extra seats remaining).
  - Then, assign the remaining seats to the states, *at most one per state*, in decreasing order of the **fractional parts** of their standard quotas (e.g., if the standard quota is 3.14159, the fractional part is 0.14159).



Alexander Hamilton
1757(?)-1804

Hamilton's method (cont'd)

- For example (p. 137-8 in R&U), use Hamilton's method apportion $h = 10$ seats to $n = 3$ states, with populations $p_1 = 1{,}450{,}000$, $p_2 = 3{,}400{,}00$, and $p_3 = 5{,}150{,}000$.

k	p_k	standard quota	lower quota	upper quota	Hamliton apportio-ment
1	1,450,000	1.45	1	2	2
2	3,400,000	3.40	3	4	3
3	5,150,000	5.15	5	6	5

Quota rule

- An apportionment method satisfies the **quota rule** (or is a **quota method**) if each state is always assigned either its upper quota or lower quota.

<u>Proposition:</u> *Hamilton's method is a quota method.*

- Why? Each state gets at least their lower quota, and since we add at most one additional seat to each state, they cannot get more than their upper quota.

- In the (extremely rare) case that the standard quota of a state is exactly an integer (and is thus equal to both upper and lower quotas), the fractional part will be exactly 0, so they will be last on a list of states in decreasing order of fractional parts. Then, all of the remaining states will have been distributed to the other states before we get to this state.

Paradoxes

- Despite its simplicity, Hamilton's method has some strange properties, particularly when the input variables (h, n, the p_k) start to change.

- For example: Consider again the case of apportioning seats to $n=3$ states, with populations $p_1 = 1{,}450{,}000$, $p_2 = 3{,}400{,}000$, and $p_3 = 5{,}150{,}000$. With $h = 10$ seats, Hamilton's method apportions 2, 3, and 5 seats respectively. What happens if we change to $h = 11$ seats?

k	p_k	standard quota	lower quota	Hamliton apportio-ment
1	1,450,000	1.595	1	1
2	3,400,000	3.740	3	4
3	5,150,000	5.665	5	6

- States 2 and 3 have gained a seat, while state 1 has lost a seat (!) despite no change in their populations.

Paradoxes (cont'd)

- This phenomena, when a state loses a seat when the size of the House is increased and all other parameters (number of states, populations) are fixed, is called the **Alabama paradox**, named for the state it affected in the 1880 re-apportionment.

- An apportionment method is **house monotone** if it avoids this paradox, i.e., if an increase in h, with all others parameters fixed, can never result in a decrease in any of the apportioned amounts a_k.

Paradoxes (cont'd)

- Now consider the case of apportioning $h = 10$ seats to $n = 3$ states, with populations $p_1 = 1{,}470{,}000$, $p_2 = 3{,}380{,}000$, and $p_3 = 4{,}650{,}000$.

k	p_k	standard quota	lower quota	Hamliton apportion-ment
1	1,470,000	1.55	1	1
2	3,380,000	3.56	3	4
3	4,650,000	4.89	5	5

- The population of state 1 increased, while the others decreased, yet state 1 lost a seat!
- This is called the **population paradox**; one state gains (or remains the same) in population while another loses (or remains the same), yet it is the first state that loses a seat, while the other gains a seat.
- Methods that avoid this are said to be **population monotone**.

Paradoxes (cont'd)

- Consider again the case of apportioning seats to states with populations $p_1 = 1{,}450{,}000$, $p_2 = 3{,}400{,}00$, and $p_3 = 5{,}150{,}000$, with the addition of a fourth state, having population $p_4 = 2{,}600{,}000$. Suppose that we wish to apportion $h = 13$ seats, since it seems appropriate to give the new state 3 seats.

k	p_k	standard quota	lower quota	Hamliton apportio-ment
1	1,450,000	1.50	1	1
2	3,400,000	3.51	3	4
3	5,150,000	5.31	5	5
4	2,600,000	2.68	2	3

- Again, state 1 has lost a seat to state 3, despite *all* of the new seats going to state 4, and their populations remaining unchanged.

Paradoxes (cont’d)

- This phenomena, when the addition of a new state causes one old state to lose a seat to another old state without a change in their populations, is called the **new states paradox**.

- This is sometimes called the **Oklahoma paradox**, since when Oklahoma became a state in 1907, and was awarded 5 new seats, Hamilton’s method would have removed a seat from New York and given it to Maine, without their population information changing.

- It can be seen as a special kind of population paradox.

Hamilton’s method

- Despite these issues, Hamilton’s method has been used for apportionment in the US, after the 1850, 1880 and 1890 censuses.
- It was not used when it was originally introduced in 1791. In fact, though Congress passed a bill supporting its use, it was vetoed (the first Presidential veto) by President George Washington, at the urging of Thomas Jefferson, on the grounds that it gave more than 1 seat per 30,000 people in some states (though not overall).
- Perhaps not coincidentally, Hamilton’s method favored New York (his state), while harming Virginia (Jefferson’s state).

- Recommended reading: Ch. 7 in R&U
- Solutions to Test 1 have been posted on the course website.