MATH 1340 – Test 1 (Voting and Social Choice)

This test has 4 **problems** (2 on the front, 2 on the back) with several parts to each. Write down all solutions in a Cornell University Official Examination Book, with your name and signature on the front cover. Make sure to label the solutions in the booklet (you do not have to do them in order). If you require an additional booklet, you may request one. This sheet can be used for scrap work only.

Show your work in all solutions! If you are showing why a property does or does not hold, give an explanation (a few sentences), or a counterexample (explain why it is a counterexample). Partial work can be worth partial credit.

You have **75 minutes** to complete this test. If you finish early, you may hand in your exam book and leave.

You may have a one-page, one-sided, hand-written set of your own notes. Calculators, laptops, tablets, cell phones, smart watches, books, and other notes are **not permitted**.

Problem 1. Consider the following profile:

В	В	С	В	D	D	А
D	D	В	А	С	С	С
А	С	Α	С	Α	В	D
С	Α	D	D	В	А	В

- (a) Determine the winners using the plurality method.
- (b) Determine the winners using the Borda count method.
- (c) Determine the winners using Copeland's method.
- (d) Is there a Condorcet candidate? An anti-Condorcet candidate? (If so, who is it?)

Problem 2. Consider the following profiles:

	В	A	С	D	В		Profile Q:		ſ	В	A	D	C	В	
Profile P:	D	В	В	С	D	1			. [С	В	В	D	С	
	C	С	Α	В	Α]			΄ Γ	D	D	Α	В	Α	
	Α	D	D	Α	С]				ſ	А	C	С	Α	D
							1	I	1		_				
	Profile B.					D	Α	С	C	D					
						В	В	В	D	B					
	Fiome n:					С	С	Α	В	C					
					Γ.	A	D	D	Α	A					
													_		

- (a) Suppose that with a certain social choice function, candidate C is the *unique* winner in both Profile P and Profile Q. What can you say about this social choice function?
- (b) Suppose that with a certain social choice function, candidate B is the *unique* winner in Profile P, and candidate C is a winner in Profile R. What can you say about this social choice function?
- (c) Suppose that with a certain social choice function, candidate A is a winner in Profile P. What can you say about this social choice function?

Problem 3.

- (a) Consider the social choice function for **two candidates** (A and B) and five voters given as follows: If a majority (i.e., 3 or more) of the *first four* voters vote for one of the candidates, then that candidate is selected as the unique winner. If exactly two of the first four voters vote for A and two vote for B, then the fifth voter (the "tie-breaker") is used to break the tie. Explain why this is functionally equivalent to the simply majority method with five voters. In particular, this method is (surprisingly) anonymous.
- (b) Consider the voting method by which potential legislation is considered in the United States Senate: There are two candidates, the bill passes (A), or it is rejected (B), and an electorate of 101 voters (100 Senators, plus the Vice President). If a majority of the Senators support the bill (i.e., vote for A), then the bill passes (i.e., A wins). If a majority of the Senators vote to reject the bill (i.e., vote for B), then it is rejected (i.e., B wins). If exactly half of the Senators vote to support the bill, while half vote to reject the bill, the Vice President's vote is used to break the tie.

Suppose that a proposal is made to remove the Vice President from the process, and instead have a 101st Senator (say, from the District of Columbia), and use the simple majority method to pass or reject legislation. Explain why this is functionally equivalent to the current method. (Ignore "political" issues, and focus on this as a social choice function.)

(c) Suppose that an alternate proposal is made to remove the Vice President from the US Senate process, and instead use an anonymous, neutral, monotone, and decisive method for passing or rejecting legislation with the existing 100 Senators. Is there such a method? If so, give an example. If not, explain why.

Problem 4.

- (a) Describe Hare's method.
- (b) Does Hare's method satisfy anonymity?
- (c) Does Hare's method satisfy the Pareto criterion?
- (d) Does Hare's method satisfy independence?
- (e) (Bonus question: attempt if you have extra time) Does Hare's method satisfy monotonicity?

Solution for 1:

- (a) B is the unique winner in the plurality method.
- (b) A gets 1 + 0 + 1 + 2 + 1 + 0 + 3 = 8 points, B gets 3 + 3 + 2 + 3 + 0 + 1 + 0 = 12 points, C gets 0 + 1 + 3 + 1 + 2 + 2 + 2 = 11 points, and D gets 2 + 2 + 0 + 0 + 3 + 3 + 1 = 11 points, so B is the unique winner in the Borda count method.
- (c) The head-to-head match-ups are as follows:

2	5		2	5		3	4		3	4		4	3		2	5
А	В	,	Α	С	,	Α	D	,	В	С	,	В	D	,	С	D
В	А		С	Α		D	Α		С	В		D	В		D	С

So, A gets 0 points, B gets 2 points, C gets 2 points, and D gets 2 points. Thus, in Copeland's method, B, C and D all win.

(d) Using the head-to-head match-ups in (c), we see that there is no Condorcet candidate, but A is the anti-Condorcet candidate.

Solution for 2:

- (a) This method fails to be neutral: Profiles P and Q are identical except that C and D have been interchanged in every preference ballot, but D fails to become a winner in Profile Q, violating neutrality.
- (b) This method fails to be independent: The relative positions of B and C are the same in Profiles P and R (voters 1, 2 and 5 prefer B to C, 3 and 4 prefer C to B), and B defeats C in Profile P, but C wins in Profile R, violating independence.
- (c) This method fails to be both Condorcet and anti-Condorcet. It fails to be Condorcet because candidate B is the Condorcet candidate (beating A 4-1, C 3-2 and D 4-1), but is not declared the unique winner. It fails to be anti-Condorcet because A is the anti-Condorcet candidate (losing to B 1-4, C 2-3, and D 2-3).

Solution for 3:

- (a) Recall that the simple majority method, for two candidates with 5 voters, will simply select as the (unique) winner whichever candidate receives at least 3 votes. In the method described above, this is also true: if one of the candidates receives at least 3 votes (either 3 from the first four voters, or two from the first four plus the tie-breaker), that candidate wins, and this is the only way for a candidate to win. Thus, this method is functionally equivalent to the simple majority method.
- (b) Recall that the simple majority method, for two candidates with 101 voters, will simply select as the (unique) winner whichever candidate receives at least 51 votes. In US Senate method, this is also true: if one of A or B receives at least 51 votes (either from 51 Senators, or from 50 Senators plus the Vice President), that candidate wins, and this is the only way for a candidate to win. Thus, the existing method is functionally equivalent to the simple majority method with 101 Senators.
- (c) There is no such method: By May's Theorem, the only anonymous, neutral, monotone and nearly decisive method for two candidates and an even (e.g., 100) sized electorate is the simple majority method, but even this fails to be decisive.

Solution for 4:

- (a) Hare's method works as follows: Given a profile, we first check if there is a candidate with a majority of first-place votes, in which case we declare them the unique winner, and otherwise we eliminate the candidate(s) with the fewest first-place vote. This yields a new profile, in which we do the same, and continue this process until either a majority winner has been found, in which case they are the unique winner, or there is an exact tie in first-place votes between all of the remaining candidate, and we declare them all to be winners.
- (b) Hare's method is anonymous since it depends only on the tabulated profiled.
- (c) Hare's method satisfies the Pareto criterion since if a candidate X is placed above a candidate Y in every preference ballot, candidate Y has no first-place votes, and thus cannot be a majority winner in the first round, and if there is no majority winner, Y is eliminated in the first round, and thus cannot be a winner.
- (d) Hare's method is not independent since by Arrow's theorem, the only method which is Pareto and independent is a dictatorship, and we have seen above that Hare's method is Pareto and not a dictatorship (since it is anonymous).
- (e) Bonus: Hare's method is not monotone: Consider the following profiles:

Before:	6	5	4	2	After:	6	5	4	2
	Α	C	В	В		А	C	В	А
	В	Α	С	Α		В	Α	С	В
	С	В	А	С		С	В	А	С

In Before, candidate C is eliminated in the first round, and A is the majority winner in the second round. In After, B is eliminated in the first round, and C is the majority winner in the second round, despite A having been moved up on the last two voters preference lists. This violates monotonicity.