# NOTES ON BILINEAR RESTRICTION

ITAMAR OLIVEIRA

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## 1. A brief description of these notes

The goal is to prove a sharp bilinear restriction theorem by Tao [3]. These notes were written as an exercise to better understand the material and no original work by the author is contained in them. The version of the theorem that we prove here is the one done in a more general setting by Lee [1]. This is because this level of generality is necessary to prove that bilinear restriction estimates imply linear ones for the paraboloid. We follow the exposition of [2] closely.

Two tools are taken for granted here: the localization theory by Tao and Vargas [4] and the wave packet decomposition (Lemma 6.1). Both can be found in Mattila's book [2].

## 2. Setting for the bilinear restriction theorem

We have $C_0, c_0, \varepsilon_0$ and $R_0$ positive constants and we have for $j = 1, 2$, bounded open sets $V_j \subset \mathbb{R}^{n-1} \subset B(0, R_0)$, $\tilde{V}_j$ is the $\varepsilon_0$-neighborhood of $V_j$, $V_j^*$ is the $4\varepsilon_0$-neighborhood of $\tilde{V}_j$, $C^2$-functions $\varphi_j : V_j^* \to \mathbb{R}$ satisfying: the maps $\nabla\varphi_j$ are diffeomorphisms such that for all $v_j \in \tilde{V}_j$, $\det(D(\nabla\varphi_j)(v_j)) \neq 0$ and

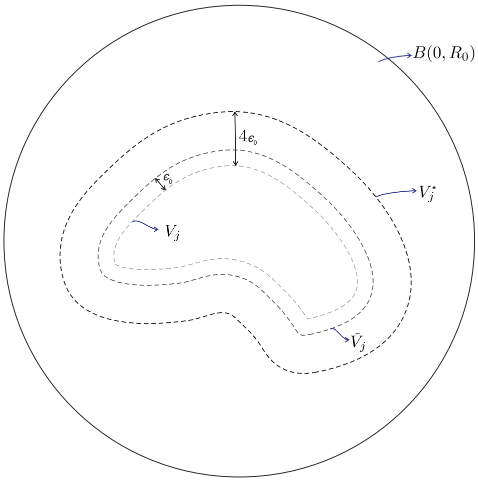
$$|\nabla\varphi_j(v_j)| \leq C_0, \tag{1}$$

$$|D(\nabla\varphi_j)(v_j)(x)| \geq c_0|x|, \quad \forall x \in \mathbb{R}^{n-1}, \tag{2}$$

$$|D(\nabla\varphi_1)(v_1)^{-1}(\nabla\varphi_2(v_2) - \nabla\varphi_1(v_1)) \cdot (\nabla\varphi_2(v_2) - \nabla\varphi_1(v_1))| \geq c_0, \tag{3}$$

$$|D(\nabla\varphi_2)(v_2)^{-1}(\nabla\varphi_1(v_1) - \nabla\varphi_2(v_2)) \cdot (\nabla\varphi_1(v_1) - \nabla\varphi_2(v_2))| \geq c_0, \tag{4}$$

$S_j = \{(x, \varphi_j(x)) : x \in V_j\}$, $j = 1, 2$, are the corresponding surfaces and $q > q_0 = \frac{n+2}{n}$.




**Claim 2.1.** *The inequalities above yield that there is $c_1 > 0$, depending only on $C_0$ and $c_0$, such that*

$$|\nabla\varphi_1(v_1) - \nabla\varphi_2(v_2)| \geq c_1 \tag{5}$$

*for all $v_1 \in \tilde{V}_1$, $v_2 \in \tilde{V}_2$.*

*Proof.* Observe that:

$$c_0 \leq |D(\nabla\varphi_1)(v_1)^{-1}(\nabla\varphi_2(v_2) - \nabla\varphi_1(v_1)) \cdot (\nabla\varphi_2(v_2) - \nabla\varphi_1(v_1)) \leq \|D(\nabla\varphi_1)^{-1}\|_\infty |\nabla\varphi_2(v_2) - \nabla\varphi_1(v_1)|^2$$

and the claim follows. ☐

Since $n_1 = (\nabla\varphi_1(v_1), 1)$ and $n_2 = (\nabla\varphi_2(v_2), 1)$ give the normal directions of the surfaces $S_1$ and $S_2$, we have $|n_1 \wedge n_2| \geq c_1$ and the surfaces are transversal.

## 3. Bilinear restriction theorem

For a function $f_j$ defined on $V_j$ we set

$$E_j f_j(x,t) = \int_{V_j} e^{2\pi i(x\cdot v + t\varphi_j(v))} f_j(v) \mathrm{d}v, \quad (x,t) \in \mathbb{R}^{n-1} \times \mathbb{R}.$$

These are the *Fourier extension operators.* Our goal is to prove the following theorem of Tao [3]:

**Theorem 3.1.** *Suppose the assumptions of Section 2 are satisfied. If $\omega$ is a non-negative weight with $\|\omega\|_\infty \leq 1$, then*

$$\|E_1 f_1 \cdot E_2 f_2\|_{L^q(\omega)} \leq C\|f_1\|_2 \|f_2\|_2 \quad \text{for } f_j \in L^2(V_j), j = 1,2.$$

**Remark 3.2.** The theorem above is stated in a more general setting than the original result of Tao.

## 4. From local to global

In this section we state a localization theorem of Tao and Vargas [4]. Assume that for $K_j \subset V_j$ compact and $\varphi_1, \varphi_2 : V_j \to \mathbb{R}$ the surfaces

$$S_j = \{(x, \varphi_j(x)) : x \in K_j\}$$

have non-vanishing Gaussian curvature. We do not need the transversality hypotheses here.

**Theorem 4.1** (Tao and Vargas). *Let $f_j \in L^2(S_j)$, $j = 1, 2$. Suppose that $\omega \in L^\infty(\mathbb{R}^n)$ with $\omega \geq 0$, $\|\omega\|_\infty \leq 1$ and $1 < p < \frac{n+1}{n-1}$. If $\alpha > 0$, $\frac{1}{p}\left(1 + \frac{4\alpha}{n-1}\right) < \frac{1}{q} + \frac{2\alpha}{n+1}$, $M_\alpha \geq 1$ and*

$$\|E_1 f_1 \cdot E_2 f_2\|_{L^q(\omega, B(x,R))} \leq M_\alpha R^\alpha \|f_1\|_{L^2(S_1)} \|f_2\|_{L^2(S_2)} \tag{6}$$

*for $x \in \mathbb{R}^n$, $R > 1$, $f_j \in L^2(S_j)$, $j = 1, 2$, then*

$$\|E_1 f_1 \cdot E_2 f_2\|_{L^p(\omega)} \leq C M_\alpha \|f_1\|_{L^2(S_1)} \|f_2\|_{L^2(S_2)}$$

*for $f_j \in L^2(S_j)$, $j = 1, 2$, where $C$ depends only on the structure constants os Section 2.*

## 5. From a local estimate to all of them: induction on scales

This crucial step is an argument due to Wolff:

**Proposition 5.1.** *Suppose that the assumptions of Section 2 are satisfied. Then there is a constant $c > 0$ such that the following holds. Assume that for some $\alpha > 0$ it holds*

$$\|E_1 f_1 \cdot E_2 f_2\|_{L^q(\omega, Q(x,R))} \leq M_\alpha R^\alpha \|f_1\|_{L^2(V_1)} \|f_2\|_{L^2(V_2)} \tag{7}$$

*for $x \in \mathbb{R}^n$, $R > 1$ and $f_j \in L^2(V_j)$, $j = 1, 2$. Then for all $0 < \delta,\ \varepsilon < 1$,*

$$\|E_1 f_1 \cdot E_2 f_2\|_{L^q(\omega, Q(a,R))} \leq C R^{\max\{\alpha(1-\delta), c\delta\}+\varepsilon} \|f_1\|_{L^2(V_1)} \|f_2\|_{L^2(V_2)}, \tag{8}$$

*for $a \in \mathbb{R}^n$, $R > 1$ and $f_j \in L^2(S_j)$, $j = 1, 2$, where the constant $C$ depends only on the structure constants of Section 2 and on $M_\alpha$, $\delta$ and $\varepsilon$.*

Notice that localizing with cubes $Q(x, R)$ (center $x$ and side-length $R$) is equivalent to doing it with balls. The point here is that once we have this proposition we can argue inductively to get down to arbitrary small $\alpha$.

**Claim 5.2.** (6) *holds for $\alpha = \alpha_0 = \frac{n}{q}$.*

*Proof.* By Cauchy-Schwarz we get $\|E_j f_j\|_\infty \lesssim \|f_j\|_{L^2(V_j)}$. Hence by Hölder:

$$\begin{aligned}\|E_1 f_1 \cdot E_2 f_2\|_{L^q(\omega, B(x,R))} &\lesssim \|\chi_{B(x,R)}\|_{L^q(\mathbb{R}^n)} \|f_1\|_{L^2(V_1)} \|f_2\|_{L^2(V_2)} \\ &\lesssim R^{\frac{n}{q}} \|f_1\|_{L^2(V_1)} \|f_2\|_{L^2(V_2)}\end{aligned}$$

for $x \in \mathbb{R}^n$, $R > 1$ and $f_j \in L^2(V_j)$, $j = 1, 2$. □

**Claim 5.3.** *Proposition 5.1 implies 6 for all $\alpha > 0$.*

*Proof.* Fix $\varepsilon > 0$ and define

$$\alpha_{j+1} = \frac{c\alpha_j}{\alpha_j + c} + \varepsilon, \quad j = 0, 1, 2, \ldots$$

By the previous claim, we know that (6) holds for some $\alpha$. Apply Proposition 5.1 with $\delta = \delta_j = \frac{\alpha_j}{\alpha_j + c}$. Then

$$\max\{\alpha_j(1-\delta), c\delta\} = \frac{c\alpha_j}{\alpha_j + c},$$

and it follows that 5.1 holds for $\alpha = \alpha_{j+1}$. It is easy to check that if $\varepsilon$ is chosen small enough, the sequence $(\alpha_j)$ is decreasing and

$$\alpha_j \to \frac{(\varepsilon + \sqrt{\varepsilon^2 + 4c\varepsilon})}{2}.$$

Since we can choose $\varepsilon$ arbitrarily small, 6 holds for all $\alpha > 0$. □

## 6. Wave packet decomposition

Set

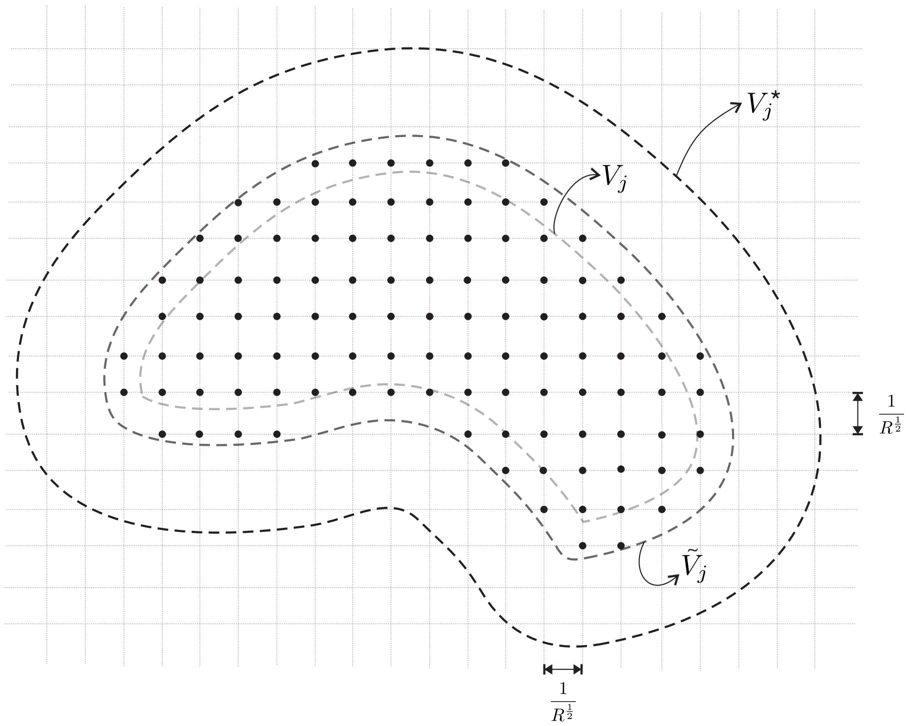
$$\mathcal{Y} = R^{\frac{1}{2}}\mathbb{Z}^{n-1},$$
$$\mathcal{V}_j = R^{-\frac{1}{2}}\mathbb{Z}^{n-1} \cap \tilde{V}_j,$$
$$\mathcal{W}_j = \mathcal{Y} \times \mathcal{V}_j.$$

For each $w_j = (y_j, v_j) \in \mathcal{W}_j$ define

$$T_{w_j} = \{(x,t) : |t| \le R, |x - (y_j - t\nabla\varphi_j(v_j))| \le R^{\frac{1}{2}}\}.$$

Then $T_{w_j}$ is a tube with center $(y_j, 0)$ and direction $(\nabla\varphi_j(v_j), 1)$. Notice that $\#\mathcal{V}_j \lesssim R^{\frac{n-1}{2}}$ and for a fixed $v_j$ the tubes $T_{y,v_j}$, $y \in \mathcal{Y}$, have bounded overlap.

Figure 1. Points in $\mathcal{V}_j$




**Lemma 6.1** (Wave-packet decomposition). *Let $C_0$ be as in Section 2. Let $f_j \in L^2(V_j)$. Then there are functions $p_{w_j} \in L^\infty(\mathbb{R}^n)$ and non-negative constants $C_{w_j}$, $w_j \in \mathcal{W}_j$, $j = 1, 2$, with the following properties for $(x,t) \in \mathbb{R}^{n-1} \times \mathbb{R}$:*

*(a)* $E_j f_j(x,t) = \sum_{w_j \in \mathcal{W}_j} C_{w_j} p_{w_j}(x,t)$.
*(b)* $p_{w_j} = E_j(\widehat{p_{w_j}(\cdot, 0)})$.
*(c)* $\|p_{w_j}\|_\infty \lesssim R^{\frac{1-n}{4}}$.
*(d)* $\operatorname{spt}(\widehat{p_{w_j}(\cdot, t)}) \subset B(v_j, 2nR^{-\frac{1}{2}})$.
*(e)* $\widehat{p_{w_j}}$ *is a measure in* $\mathcal{M}(\mathbb{R}^n)$ *with*

$$\operatorname{spt}(\widehat{p_{w_j}}) \subset S_j \cap \{(x,t) : |x - v_j| \le 2nR^{-\frac{1}{2}}\} \subset B((v_j, \varphi(v_j)), 2n(1 + C_0)R^{-\frac{1}{2}}).$$

*(f)* $\sum_{w_j \in \mathcal{W}_j} |C_{w_j}|^2 \lesssim \|f_j\|_2^2$.

*(g) If $L$ is a sufficiently large constant and $|t| \le R$ or $|x - (y_j - t\nabla\varphi_j(v_j))| > LR^{-\frac{1}{2}}|t|$, then*

$$|p_{w_j}(x,t)| \lesssim_N R^{\frac{1-n}{4}} \left(1 + \frac{|x - (y_j - t\nabla\varphi_j(v_j))|}{\sqrt{R}}\right)^{-N} \quad \text{for all } N \in \mathbb{N}.$$

*In particular, if $|t| \le R$ and $\lambda \ge 1$,*

$$|p_{w_j}(x,t)| \lesssim_\delta R^{-10n} \quad \text{if} \quad d((x,t), T_{w_j}) \ge R^{\delta + \frac{1}{2}},$$

$$|p_{w_j}(x,t)| \lesssim (\lambda R)^{-10n} \quad \text{if} \quad d((x,t), T_{w_j}) \ge \lambda R.$$

*(h) If $|t| \le R$, then for any $W \subset \mathcal{W}_j$,*

$$\left\| \sum_{w_j \in W} p_{w_j}(\cdot, t) \right\|_2^2 \lesssim \#W.$$

*(i) The product $p_{w_1} p_{w_2} \in L^2(\mathbb{R}^n)$.*

## 7. Several reductions through dyadic pigeonholing

We now begin the proof of Proposition 5.1. By Claim 5.2, we have a base case: there is $\alpha > 0$ for which (6) holds. Fix $R > 1$, which we can choose later as big as we want. To prove (8) we may assume $a = 0$, $nR^{-\frac{1}{2}} < \varepsilon_0$ and fix $f_j \in L^2(V_j)$ with $\|f_j\|_2 = 1$ for $j = 1, 2$.

By the Lemma 6.1, it suffices to prove that

$$\left\| \sum_{w_1 \in \mathcal{W}_1} \sum_{w_2 \in \mathcal{W}_2} C_{w_1} C_{w_2} p_{w_1} p_{w_2} \right\|_{L^q(\omega, Q(R))} \lesssim R^\varepsilon (R^{\alpha(1-\delta)} + R^{c\delta}), \tag{9}$$

for some positive constant $c$, where $Q(R)$ is the cube in $\mathbb{R}^n$ with centre 0 and side-length $R$. Below $c$ will always depend on the setting described in Section 2, but we will often increase its value while going on.

We are now going to describe several reductions that imply estimate (9). These reductions are of two kinds: we first get rid of many tubes in the left-hand side of (9) that make irrelevant contributions to the $L^q$ norm. The other kind is known as *dyadic pigeonholing*.

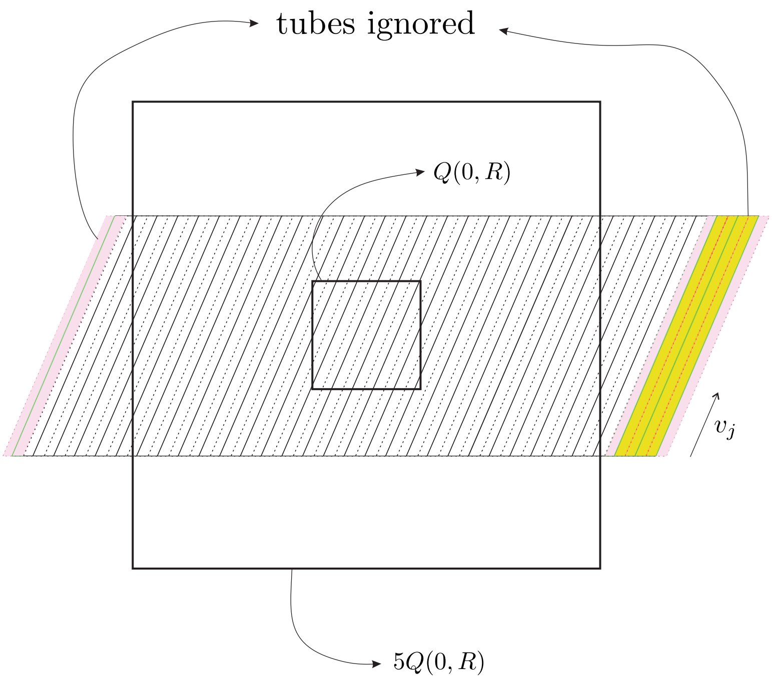
(1) **First reduction:** It is enough to consider $w_j$ for which $T_{w_j} \cap 5Q(R) \neq \emptyset$ for $j = 1, 2$.
Indeed, split the sum

$$\sum_{w_1 \in \mathcal{W}_1} \sum_{w_2 \in \mathcal{W}_2} = \sum_{\textbf{(A)}} + \sum_{\textbf{(B)}} + \sum_{\textbf{(C)}} + \sum_{\textbf{(D)}} \quad \text{where}$$

- **(A)**: $\{(w_1, w_2) \in \mathcal{W}_1 \times \mathcal{W}_2; T_{w_j} \cap 5Q(R) \neq \emptyset, j = 1, 2\}$
- **(B)**: $\{(w_1, w_2) \in \mathcal{W}_1 \times \mathcal{W}_2; T_{w_1} \cap 5Q(R) = \emptyset$ and $T_{w_2} \cap 5Q(R) \neq \emptyset\}$
- **(C)**: $\{(w_1, w_2) \in \mathcal{W}_1 \times \mathcal{W}_2; T_{w_2} \cap 5Q(R) = \emptyset$ and $T_{w_1} \cap 5Q(R) \neq \emptyset\}$
- **(D)**: $\{(w_1, w_2) \in \mathcal{W}_1 \times \mathcal{W}_2; T_{w_1} \cap 5Q(R) = \emptyset$ and $T_{w_2} \cap 5Q(R) = \emptyset\}$.

We show that **(B)** is irrelevant (and so are **(C)** and **(D)** by the same kind of argument). By *(c)* and *(f)* of Lemma 6.1, $|C_{w_1} C_{w_2}| \lesssim 1$. The cardinality of $w_2 \in \mathcal{W}_2$ such that $T_{w_2} \cap 5Q(R) \neq \emptyset$ is roughly $R^{n-1}$ since for $w_2 = (y, v)$ we have $\approx \frac{R^n}{R^{\frac{n-1}{2}} \times R} = R^{\frac{n-1}{2}}$ choices of $y$ and $R^{\frac{n-1}{2}}$ of $v$.

The number of $w_1 \in \mathcal{W}_1$ such that $5^k R < d(T_{w_1}, Q(R)) \le 5^{k+1} R$ is dominated by $\approx R^{\frac{n-1}{2}} \times (5^k R^{\frac{1}{2}})^{n-1} = (5^k R)^{n-1}$. Thus using *(g)* of Lemma 6.1 we get




$$\begin{aligned}\Big\| \sum_{\mathbf{(B)}} C_{w_1} C_{w_2} p_{w_1} p_{w_2} \Big\|_{L^q(\omega, Q(R))} &\lesssim R^{n-1} \sum_{k=0}^{\infty} \sum_{\substack{w_1 \in \mathcal{W}_1 \\ 5^k R < d(T_{w_1}, Q(R)) \leq 5^{k+1} R}} \|p_{w_1}\|_{L^q(\omega, Q(R))} \\ &\lesssim \sum_{k=0}^{\infty} R^n \#\{w_1 \in \mathcal{W}_1 : 5^k R < d(T_{w_1}, Q(R)) \leq 5^{k+1} R\} (5^k R)^{-10n} (5^k R)^{\frac{n}{q}} \\ &\lesssim \sum_{k=0}^{\infty} 5^{-k} R^{-7n} < 2R^{-7n}.\end{aligned}$$

(2) **<u>Second reduction:</u>** We can assume that for some constant $C$,

$$R^{-10n} \leq C_{w_j} \leq C.$$

Indeed, the number of pairs $(w_1, w_2) \in \mathbf{(A)}$ is $\lesssim R^{2(n-1)}$, so

$$\sum_{\substack{\mathbf{(A)} \\ |C_{w_1}| \leq R^{-10n} \text{ or } |C_{w_2}| \leq R^{-10n}}} \|C_{w_1} C_{w_2} p_{w_1} p_{w_2}\|_{L^q(w, Q(R))} \lesssim R^{-8n}.$$

From now on we replace the sets $\mathcal{W}_j$ by their subsets which correspond to those $w_j$ for which the conditions in these first two reductions are satisfied.

(3) **<u>Third reduction:</u>** We get rid of the $C_{w_j}$ by dyadic pigeonholing. There are about $\log R$ dyadic numbers in $[R^{-10n}, C]$, so by the triangle inequality

$$\left\|\sum_{w_1\in\mathcal{W}_1}\sum_{w_2\in\mathcal{W}_2} C_{w_1}C_{w_2}p_{w_1}p_{w_2}\right\|_{L^q(w,Q(R))} \leq \sum_{\substack{\kappa_1,\kappa_2\in[R^{-10n},C]\\ \kappa_1,\kappa_2 \text{ dyadic}}} \left\|\sum_{\substack{w_1\in\mathcal{W}_1\\ \kappa_1<C_{w_1}\leq 2\kappa_1}}\sum_{\substack{w_2\in\mathcal{W}_2\\ \kappa_2<C_{w_2}\leq 2\kappa_2}} C_{w_1}C_{w_2}p_{w_1}p_{w_2}\right\|_{L^q(w,Q(R))}$$

$$\lesssim (\log R)^2 \left\|\sum_{\substack{w_1\in\mathcal{W}_1\\ \kappa_1<C_{w_1}\leq 2\kappa_1}}\sum_{\substack{w_2\in\mathcal{W}_2\\ \kappa_2<C_{w_2}\leq 2\kappa_2}} C_{w_1}C_{w_2}p_{w_1}p_{w_2}\right\|_{L^q(w,Q(R))}$$

for some pair $(\kappa_1, \kappa_2)$, namely the one that maximizes the inner norm in the first line of the right-hand side above. Writing $\tilde{p}_{w_j} = \left(\frac{C_{w_j}}{\kappa_j}\right) p_{w_j}$ and $W_j$ for the set of $w_j \in \mathcal{W}_j$ for which $\kappa_j \leq C_{w_j} \leq 2\kappa_j$ we have

$$\left\|\sum_{\substack{w_1\in\mathcal{W}_1\\ \kappa_1<C_{w_1}\leq 2\kappa_1}}\sum_{\substack{w_2\in\mathcal{W}_2\\ \kappa_2<C_{w_2}\leq 2\kappa_2}} C_{w_1}C_{w_2}p_{w_1}p_{w_2}\right\|_{L^q(w,Q(R))} = \left\|\sum_{w_1\in W_1}\sum_{w_2\in W_2} \tilde{p}_{w_1}\tilde{p}_{w_2}\right\|_{L^q(w,Q(R))} \kappa_1\kappa_2.$$

Since by *(f)* of Lemma 6.1 $\sqrt{\#W_j} \lesssim \frac{1}{\kappa_j}$ and since the functions $\tilde{p}_{w_j}$ satisfy all the conditions $(b)-(e)$ and $(g)-(i)$ (we will not use *(a)* or *(f)* anymore), it suffices to show that

$$\left\|\sum_{w_1\in W_1}\sum_{w_2\in W_2} p_{w_1}p_{w_2}\right\|_{L^q(w,Q(R))} \lesssim R^{\varepsilon}(R^{\alpha(1-\delta)} + R^{c\delta})\sqrt{\#W_1\#W_2},$$

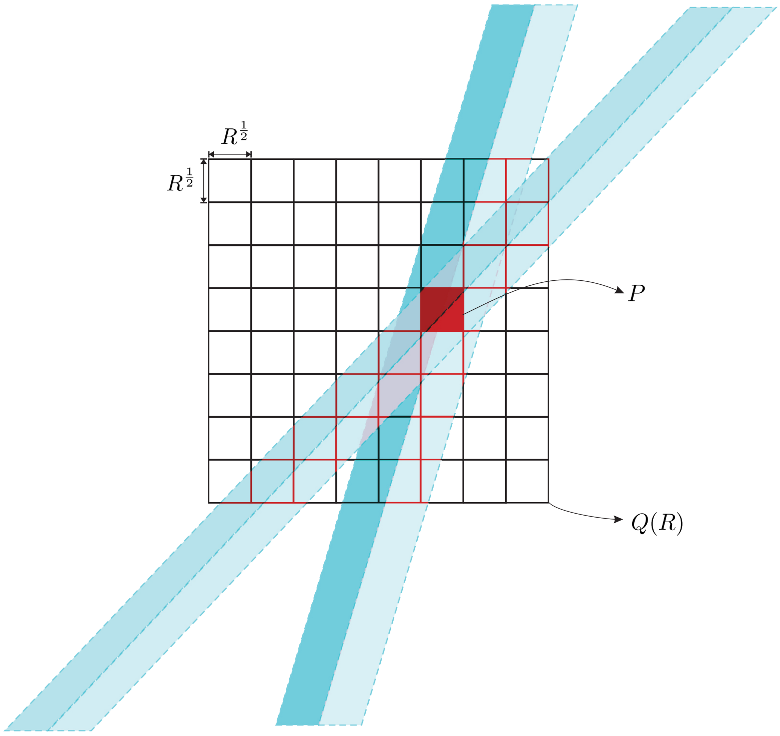
for $W_1 \subset \mathcal{W}_1$, $W_2 \subset \mathcal{W}_2$, $p_{w_j}$ satisfying all conditions but *(a)* and *(f)* from Lemma 6.1, $j = 1, 2$.

(4) **Fourth reduction:** We will use the induction hypotheses (7) to reduce the estimate above once more. Decompose $Q(R)$ into $R^{\frac{n}{2}}$ cubes $P \in \mathcal{P}$ of side-length $\sqrt{R}$ (assuming that $\sqrt{R}$ is too an integer). For $P \in \mathcal{P}$, set

$$W_j(P) = \{w_j \in W_j : T_{w_j} \cap R^{\delta}P \neq \emptyset\}.$$

Observe that $\#W_j(P)$ essentially counts how many of the tubes parametrized by $W_j$ intersect $P$.

Figure 2. In the picture below, the points of $W_j(P)$ represent 4 tubes.




For dyadic integers $1 \leq \kappa_1, \kappa_2 \leq R^{2n}$ set

$$\mathcal{Q}(\kappa_1, \kappa_2) = \{P \in \mathcal{P} : \kappa_1 < \#W_1(P) \leq 2\kappa_1, \kappa_2 < \#W_2(P) \leq 2\kappa_2\},$$

and for $w_j \in W_j$,

$$\lambda(w_j, \kappa_1, \kappa_2) = \#\{P \in \mathcal{Q}(\kappa_1, \kappa_2) : T_{w_j} \cap R^\delta P \neq \emptyset\},$$

and for dyadic integers $1 \leq \lambda \leq R^{2n}$,

$$W_j(\lambda, \kappa_1, \kappa_2) = \{w_j \in W_j : \lambda < \lambda(w_j, \kappa_1, \kappa_2) \leq 2\lambda\}.$$

In other words, $\lambda(w_j, \kappa_1, \kappa_2)$ essentially counts how many cubes in $\mathcal{Q}(\kappa_1, \kappa_2)$ intersect the tube $T_{w_j}$, and $W_j(\lambda, \kappa_1, \kappa_2)$ essentially represents the set of tubes that intersect $\approx \lambda$ cubes of $\mathcal{Q}(\kappa_1, \kappa_2)$.

Decompose $Q(R)$ once more (this decomposition is independent of the one we just did) into $R^{\delta n}$ cubes $Q \in \mathcal{Q}$ of side-length $R^{1-\delta}$ (assume $R^\delta$ integer without loss of generality). Then

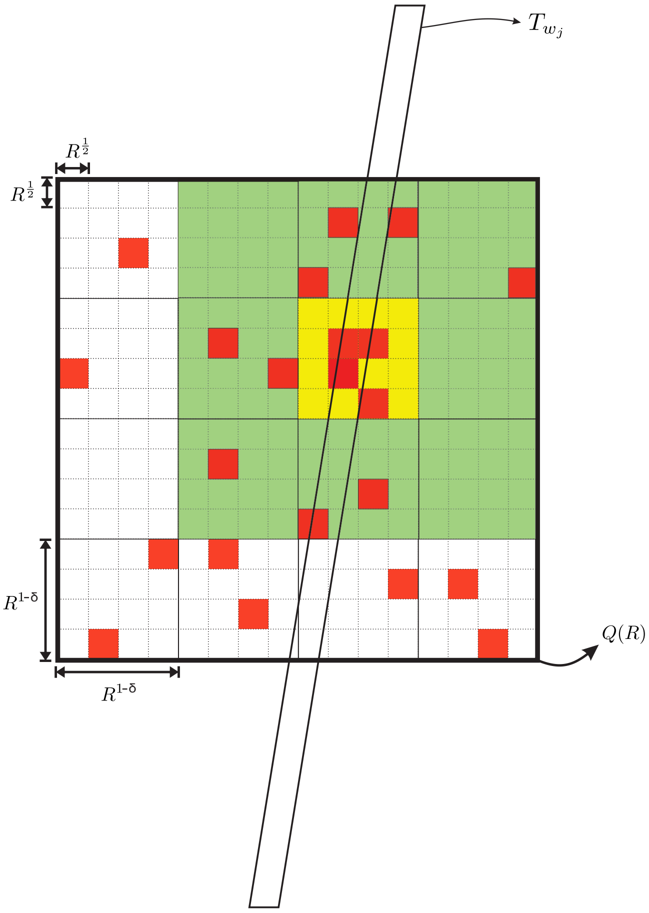
$$\left\| \sum_{w_1 \in W_1} \sum_{w_2 \in W_2} p_{w_1} p_{w_2} \right\|_{L^q(w, Q(R))} \lesssim \sum_{Q \in \mathcal{Q}} \left\| \sum_{w_1 \in W_1} \sum_{w_2 \in W_2} p_{w_1} p_{w_2} \right\|_{L^q(w, Q)}. \tag{10}$$

For dyadic integers $1 \leq \lambda, \kappa_1, \kappa_2 \leq R^{2n}$ and $w_j \in W_j(\lambda, \kappa_1, \kappa_2)$ we choose a cube $Q(w_j, \lambda, \kappa_1, \kappa_2) \in \mathcal{Q}$ which maximizes the quantity

$$\#\{P \in \mathcal{Q}(\kappa_1, \kappa_2) : T_{w_j} \cap R^\delta P \neq \emptyset, P \cap Q \neq \emptyset\}$$

among the cubes $Q \in \mathcal{Q}$. ***In other words, we choose the cube*** $Q \in \mathcal{Q}$ ***that intersects the tube*** $T_{w_j}$ ***and intersects (contains) the biggest number of cubes in*** $\mathcal{Q}(\kappa_1, \kappa_2)$.

Figure 3. The cubes of $\mathcal{Q}(\kappa_1,\kappa_2)$ are in red. Four cubes $P \in \mathcal{Q}(\kappa_1,\kappa_2)$ in the yellow cube of side length $R^{1-\delta}$ satisfy $T_{w_j} \cap R^\delta P \neq \emptyset$, and this is the maximum possible in the picture. Therefore, if $w_j \in W_j(\lambda,\kappa_1,\kappa_2)$, the yellow cube is $Q(w_j,\lambda,\kappa_1,\kappa_2)$.




Since $\#\mathcal{Q} = R^{n\delta}$, it follows that

$$\#\{P \in \mathcal{Q}(\kappa_1,\kappa_2) : T_{w_j} \cap R^\delta P \neq \emptyset, P \cap Q(w_j,\lambda,\kappa_1,\kappa_2) \neq \emptyset\} \geq \lambda R^{-n\delta}. \tag{11}$$

Indeed, if had $<$ instead of $\geq$ in the inequality above, then summing both sides over $Q \in \mathcal{Q}$ would imply $\lambda(w_j,\kappa_1,\kappa_2) < \lambda$, which contradicts $w_j \in W_j(\lambda,\kappa_1,\kappa_2)$.

We define for $w_j \in W_j$ and $Q \in \mathcal{Q}$:

$$\boxed{w_j \sim Q \iff Q \cap 10Q(w_j,\lambda,\kappa_1,\kappa_2) \neq \emptyset \quad \text{for some dyadic integers } \lambda,\kappa_1,\kappa_2 \in [1,R^{2n}].}$$

***The cubes of radius $R^{1-\delta}$ that are painted in the picture above (yellow and green) are all related by $\sim$ to $w_j$. For the picture to be more precise, we would have to have painted in green more squares around the yellow one. However, this one gives the right intuition: we have a special $Q(w_j,\lambda,\kappa_1,\kappa_2)$ and in a small neighborhood of it we have $O(1)$ other $R^{1-\delta}$-sided cubes there are also related to $w_j$.***

There are roughly $(\log R)^3$ dyadic triples $(\lambda,\kappa_1,\kappa_2) \in [1,R^{2n}]^3$, so for all $w_j \in W_j$,

$$\#\{Q \in \mathcal{Q} : w_j \sim Q\} \lesssim R^\varepsilon.$$

Thus by $(b)$ and $(h)$ of Lemma 6.1, the induction hypotheses (7) (recall that $Q$ is an $R^{1-\delta}$ cube), Plancherel's theorem and Cauchy-Schwarz:

$$\begin{aligned}
\sum_{Q\in\mathcal{Q}}\left\|\sum_{\substack{w_1\in W_1\\ w_1\sim Q}}\sum_{\substack{w_2\in W_2\\ w_2\sim Q}}p_{w_1}p_{w_2}\right\|_{L^q(w,Q)} &\leq \sum_{Q\in\mathcal{Q}}\left\|E_1\left(\sum_{\substack{w_1\in W_1\\ w_1\sim Q}}\widehat{p_{w_1}(\cdot,0)}\right)E_2\left(\sum_{\substack{w_2\in W_2\\ w_2\sim Q}}\widehat{p_{w_2}(\cdot,0)}\right)\right\|_{L^q(w,Q)}\\
&\lesssim R^{\alpha(1-\delta)}\sum_{Q\in\mathcal{Q}}\left\|\sum_{\substack{w_1\in W_1\\ w_1\sim Q}}p_{w_1}(\cdot,0)\right\|_2\left\|\sum_{\substack{w_2\in W_2\\ w_2\sim Q}}p_{w_2}(\cdot,0)\right\|_2\\
&\lesssim R^{\alpha(1-\delta)}\sum_{Q\in\mathcal{Q}}(\#\{w_1\in W_1: w_1\sim Q\})^{\frac{1}{2}}(\#\{w_2\in W_2: w_2\sim Q\})^{\frac{1}{2}}\\
&\lesssim R^{\alpha(1-\delta)}\left(\sum_{w_1\in W_1}\#\{Q\in\mathcal{Q}: w_1\sim Q\}\right)^{\frac{1}{2}}\left(\sum_{w_2\in W_2}\#\{Q\in\mathcal{Q}: w_2\sim Q\}\right)^{\frac{1}{2}}\\
&\lesssim R^{\varepsilon}R^{\alpha(1-\delta)}(\#W_1)^{\frac{1}{2}}(\#W_2)^{\frac{1}{2}}.
\end{aligned}$$

Recalling (10), it is enough to prove

$$\sum_{Q\in\mathcal{Q}}\left\|\sum_{\substack{w_1\in W_1,w_2\in W_2\\ w_1\not\sim Q\text{ or }w_2\not\sim Q}}p_{w_1}p_{w_2}\right\|_{L^q(w,Q)}\lesssim R^{c\delta}(\#W_1)^{\frac{1}{2}}(\#W_2)^{\frac{1}{2}}.$$

Since $\#Q=R^{n\delta}$, it suffices to show that for all $Q\in\mathcal{Q}$,

$$\left\|\sum_{\substack{w_1\in W_1,w_2\in W_2\\ w_1\not\sim Q\text{ or }w_2\not\sim Q}}p_{w_1}p_{w_2}\right\|_{L^q(w,Q)}\lesssim R^{c\delta}(\#W_1)^{\frac{1}{2}}(\#W_2)^{\frac{1}{2}}. \tag{12}$$

***In other words, each tube $T_{w_j}$ is allowed to exclude $\lesssim R^{\varepsilon}$ cubes of $\mathcal{Q}$ that intersect it. In the literature this is described as a "local" estimate, in the sense that for a given $w_j$ the cubes $Q$ with $w_j\sim Q$ are contained in some cube with side length $\approx R^{1-\delta}$, and to deal with it we use the induction hypothesis.***

## 8. An $L^2$ reduction

**Claim 8.1.** *It suffices to prove*

$$\left\|\sum_{\substack{w_1\in W_1,w_2\in W_2\\ w_1\not\sim Q\text{ or }w_2\not\sim Q}}p_{w_1}p_{w_2}\right\|_{L^2(Q)}\lesssim R^{c\delta-\frac{(n-2)}{4}}(\#W_1)^{\frac{1}{2}}(\#W_2)^{\frac{1}{2}}. \tag{13}$$

*Proof.* We show that (13) implies (12). Decompose the sum into the parts $w_1\not\sim Q$ and $w_2\not\sim Q$, $w_1\sim Q$ and $w_2\not\sim Q$, and $w_1\not\sim Q$ and $w_2\sim Q$. They can all be treated in the same way and we consider only the first one. By Cauchy-Schwarz and $(h)$ of Lemma 6.1,

$$\begin{aligned}\Bigg\|\sum_{\substack{w_1\in W_1,w_2\in W_2\\ w_1\not\sim Q \text{ and } w_2\not\sim Q}} p_{w_1}p_{w_2}\Bigg\|_{L^1(Q)} &\lesssim \Bigg\|\sum_{\substack{w_1\in W_1\\ w_1\not\sim Q}} p_{w_1}\Bigg\|_{L^2(Q)} \Bigg\|\sum_{\substack{w_2\in W_2\\ w_2\not\sim Q}} p_{w_2}\Bigg\|_{L^2(Q)}\\
&\lesssim \left(\int_{-R}^{R}\int_{\mathbb{R}^{n-1}}\Bigg|\sum_{\substack{w_1\in W_1\\ w_1\not\sim Q}} p_{w_1}(x,t)\Bigg|^2 \mathrm{d}x\mathrm{d}t\right)^{\frac12}\left(\int_{-R}^{R}\int_{\mathbb{R}^{n-1}}\Bigg|\sum_{\substack{w_2\in W_2\\ w_2\not\sim Q}} p_{w_2}(x,t)\Bigg|^2 \mathrm{d}x\mathrm{d}t\right)^{\frac12}\\
&\lesssim R(\#W_1)^{\frac12}(\#W_2)^{\frac12}.\end{aligned}$$

Thus we have the $L^1$ estimate

$$\Bigg\|\sum_{\substack{w_1\in W_1,w_2\in W_2\\ w_1\not\sim Q \text{ or } w_2\not\sim Q}} p_{w_1}p_{w_2}\Bigg\|_{L^1(Q)} \lesssim R(\#W_1)^{\frac12}(\#W_2)^{\frac12}.$$

This estimate and the inequality $\|g\|_q \le \|g\|_2^{2\frac{(q-1)}{q}}\|g\|_1^{\frac{(2-q)}{q}}$ yields (12).

□

For (13) it is enough to show that

$$\sum_{P\in\mathcal{P},P\subset 2Q}\int_P\Bigg|\sum_{\substack{w_1\in W_1,w_2\in W_2\\ w_1\not\sim Q \text{ or } w_2\not\sim Q}} p_{w_1}p_{w_2}\Bigg|^2 \lesssim R^{c\delta-\frac{(n-2)}{2}}\#W_1\#W_2.$$

If $w_j\notin W_j(P)$ for $j=1$ or $j=2$, then $|p_{w_1}p_{w_2}|\lesssim R^{-10n}$ on $P$ by parts $(c)$ and $(g)$ of Lemma 6.1. Since $\#(W_1\times W_2)\lesssim R^{2n}$, we have on $P$,

$$\sum_{\substack{w_1\in W_1,w_2\in W_2\\ w_1\notin W_1(P) \text{ or } w_2\notin W_2(P)}} |p_{w_1}p_{w_2}|\lesssim R^{-8n}.$$

Writing

$$\sum_{\substack{w_1\in W_1,w_2\in W_2\\ w_1\not\sim Q \text{ or } w_2\not\sim Q}} p_{w_1}p_{w_2} = g+h,$$

where $g$ consists of the terms for which $w_j\in W_j(P)$ for $j=1$ and $j=2$ and $h$ consists of the rest, we have $|g|\lesssim R^{2n}$, $|h|\lesssim R^{-8n}$ on $P$ and

$$\int_P|g+h|^2\le\int_P|g|^2+2\int_P|gh|+\int_P|h|^2\lesssim\int_P|g|^2+R^{-5n}.$$

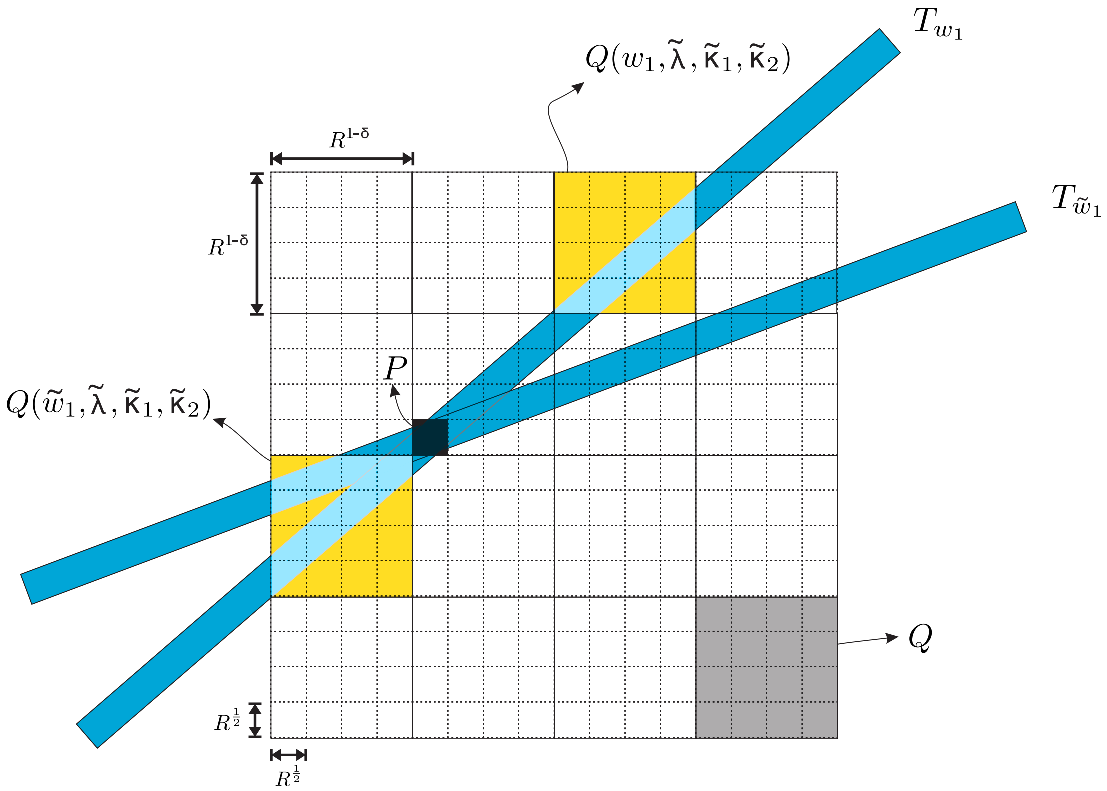
There are $\lesssim R^n$ cubes $P\in\mathcal{P}$ with $P\subset 2Q$, so it suffices to show that

$$\sum_{P\in\mathcal{P},P\subset 2Q}\int_P\Bigg|\sum_{\substack{w_1\in W_1(P),w_2\in W_2(P)\\ w_1\not\sim Q \text{ or } w_2\not\sim Q}} p_{w_1}p_{w_2}\Bigg|^2 \lesssim R^{c\delta-\frac{(n-2)}{2}}\#W_1\#W_2.$$

Pigeonholing as before we can reduce to sum over $P \in Q(\kappa_1, \kappa_2)$, $P \subset 2Q$, for some dyadic integers $\kappa_1, \kappa_2 \in [1, R^{2n}]$. By further pigeonholing we can replace $W_j(P)$ by $W_j(P) \cap W_j(\lambda_j, \kappa_1, \kappa_2)$ for some dyadic integers $\lambda_1, \lambda_2 \in [1, R^{2n}]$. Let us put

$$W_j^{\nsim Q}(P, \lambda, \kappa_1, \kappa_2) = \{w_j \in W_j(P) \cap W_j(\lambda, \kappa_1, \kappa_2) : w_j \nsim Q\},$$

Figure 4. For $w_1, \tilde{w}_1 \in W_j(P) \cap W_j(\lambda, \kappa_1, \kappa_2)$ we painted the corresponding cubes $Q(w_1, \tilde{\lambda}, \tilde{\kappa_1}, \tilde{\kappa_2})$ and $Q(\tilde{w}_1, \tilde{\lambda}, \tilde{\kappa_1}, \tilde{\kappa_2})$ for some dyadic numbers $\tilde{\lambda}, \tilde{\kappa_1}, \tilde{\kappa_2}$. Roughly speaking, in the definition of $W_1^{\nsim Q}(P, \lambda, \kappa_1, \kappa_2)$ we are collecting all tubes in $W_j(P) \cap W_j(\lambda, \kappa_1, \kappa_2)$ such that there are no dyadic numbers $\tilde{\lambda}, \tilde{\kappa_1}, \tilde{\kappa_2}$ for which the "special" squares associated to the tubes and to these parameters intersect $Q$.




and for $U_j \subset W_j$,

$$U_j(P) = \{w_j \in U_j : T_{w_j} \cap R^{\delta} P \neq \emptyset\}.$$

Breaking the sum over $w_1 \nsim Q$ or $w_2 \nsim Q$ into three sums over $w_1 \nsim Q$ and $w_2 \nsim Q$, $w_1 \nsim Q$ and $w_2 \sim Q$, and $w_1 \sim Q$ and $w_2 \nsim Q$, it is enough to show that for all $Q \in \mathcal{Q}$, any $U_2 \subset W_2$ and any dyadic integers $1 \leq \lambda, \kappa_1, \kappa_2 \leq R^{2n}$,

$$\sum_{P \in Q(\kappa_1, \kappa_2), P \subset 2Q} \int_P \left| \sum_{w_1 \in W_1^{\nsim Q}(P, \lambda, \kappa_1, \kappa_2)} \sum_{w_2 \in U_2(P)} p_{w_1} p_{w_2} \right|^2 \lesssim R^{c\delta - \frac{(n-2)}{2}} \# W_1 \# W_2. \tag{14}$$

## 9. Exploiting the geometry of the paraboloid

***Before we proceed, let us discuss some heuristics of the particular case where the surfaces are disjoint caps on the paraboloid $y = |x|^2$. Suppose that we wish to estimate a quantity of the form***

$$\left\| \sum_{w_1 \in W_1} \sum_{w_2 \in W_2} p_{w_1} p_{w_2} \right\|_2^2 . \tag{15}$$

***Ignore for now the region of spacetime we are integrating over. We can expand this expression as***

$$\sum_{w_1 \in W_1} \sum_{w_2 \in W_2} \sum_{w_1' \in W_1} \sum_{w_2' \in W_2} \langle p_{w_1} p_{w_2}, p_{w_1'} p_{w_2'} \rangle .$$

.

***By $(e)$ of Lemma 6.1, $p_{w_1}$ is supported near $(v_1, |v_1|^2)$ (similarly for $T_2$, $T_1'$ and $T_2'$). From Parseval's formula, we thus expect the above inner product to be very small unless $v_1 + v_2$ is close to $v_1' + v_2'$ and $|v_1|^2 + |v_2|^2$ is close to $|v_1'|^2 + |v_2'|^2$.***

***Suppose that we fix two of the directions, say $v_1$ and $v_1'$. Then the relation $v_1 + v_2 = v_1' + v_2'$ will correlate $v_2$ and $v_2'$, in the sense that either of these two velocities will determine the other. By exploiting these two constraints in a very fine way, Tao was able to finish the proof.***

***Just to sketch what is coming up, define***

$$\Omega_1 := \left\{ v \in \mathbb{R}^{n-1} : |v - e_1| \lesssim 1 \right\},$$

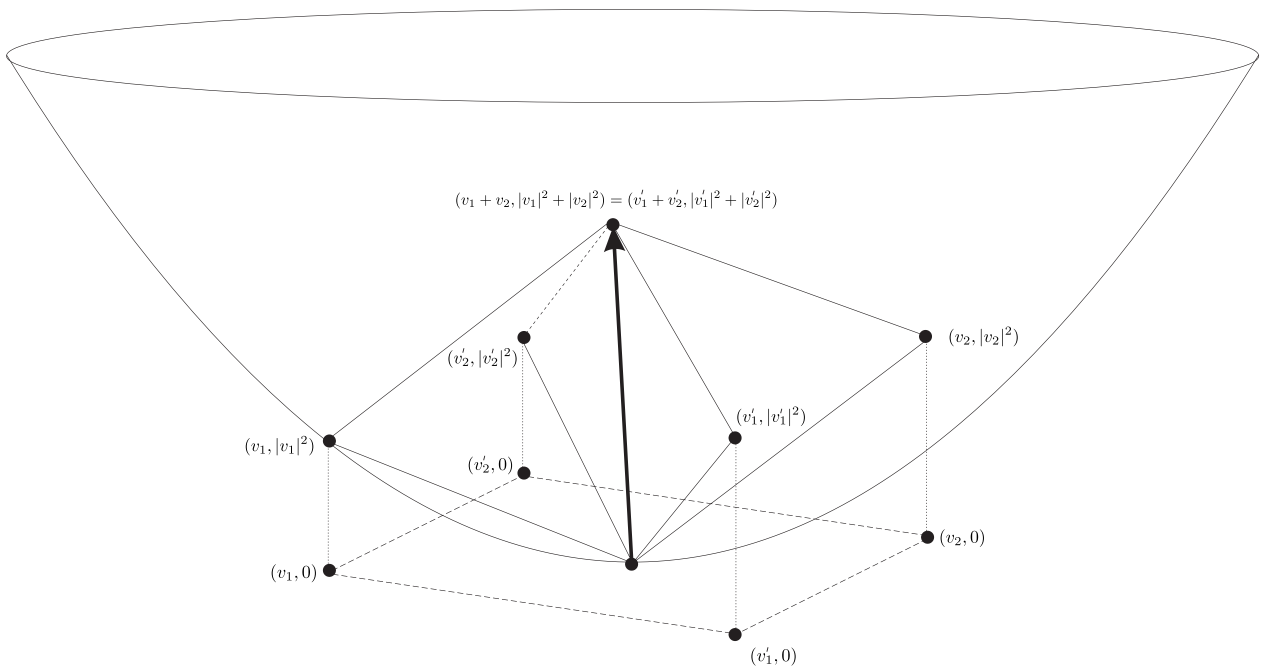
$$\Omega_2 := \left\{ v \in \mathbb{R}^{n-1} : |v + e_1| \lesssim 1 \right\},$$

***and supose that the lifts of these regions to the paraboloid are slightly larger than the original caps. For any $v_1 \in \Omega_1$, $v_2' \in \Omega_2$, let***

$$\Pi_{v_1, v_2'} := \{ v_1' \in \Omega_1 : v_1 + v_2 = v_1' + v_2', |v_1|^2 + |v_2|^2 = |v_1'|^2 + |v_2'|^2 \text{ for some } v_2 \in \Omega_2 \}.$$

***One can interpret this set as being equivalent to the set of all parallelograms with two vertices in (a slight enlargement of) each one of the original caps. A little algebra shows that $\Pi_{v_1, v_2'}$ is contained in the $n-2$-dimensional hyperplane of $\mathbb{R}^{n-1}$ which contains $v_1$ and is orthogonal to $v_2' - v_1$ (develop $|v_1 + v_2|^2 = |v_1' + v_2'|^2$ to conclude that). In other words,***

$$\left\langle v_1' - v_1, v_2' - v_1 \right\rangle = 0 \text{ whenever } v_1' \in \Pi_{v_1, v_2'} .$$




***This orthogonality is not absolutely essential to the argument; what is important is that the set $\Pi_{v_1,v_2'}$ is contained in a hypersurface with is transverse to $v_2' - v_1$, or indeed to any vector in $\Omega_2 - \Omega_1$.***

---

Now we go back to the more general setting with $\varphi_1$ and $\varphi_2$ instead of the paraboloid. Covering $V_j$ with finitely many small cubes of diameter at most $\varepsilon_0$, we may assume that $V_j$ is such a cube (write $E_1 f_1 E_2 f_2$ as a finite sum of similar expressions where the input functions are defined on such cubes). In this new setting we have:

$$v_1' + v_2' - v_1 \in V_2^* \quad \text{whenever} \quad v_1, v_1' \in \tilde{V}_1 \quad \text{and } v_2' \in \tilde{V}_2,$$

and vice versa with respect to $V_1$ and $V_2$. Moreover, when these cubes are sufficiently small

$$|\nabla\varphi_j(v_j') - \nabla\varphi_j(v_j)| < \varepsilon_1 \quad \text{whenever } v_j, v_j' \in \tilde{V}_j, \tag{16}$$

where $\varepsilon_1$ is a small constant that will be specified later. Define for $v_1 \in \tilde{V}_1$, $v_2' \in \tilde{V}_2$,

$$\Phi_{v_1,v_2'} : V_1 \to \mathbb{R}, \quad \Phi_{v_1,v_2'}(v_1') = \varphi_1(v_1) + \varphi_2(v_1' + v_2' - v_1) - \varphi_1(v_1') - \varphi_2(v_2'),$$

and

$$\Pi_{v_1,v_2'} = \{v_1' \in \tilde{V}_1 : \Phi_{v_1,v_2'}(v_1') = 0\}.$$

By the setting of Section 2,

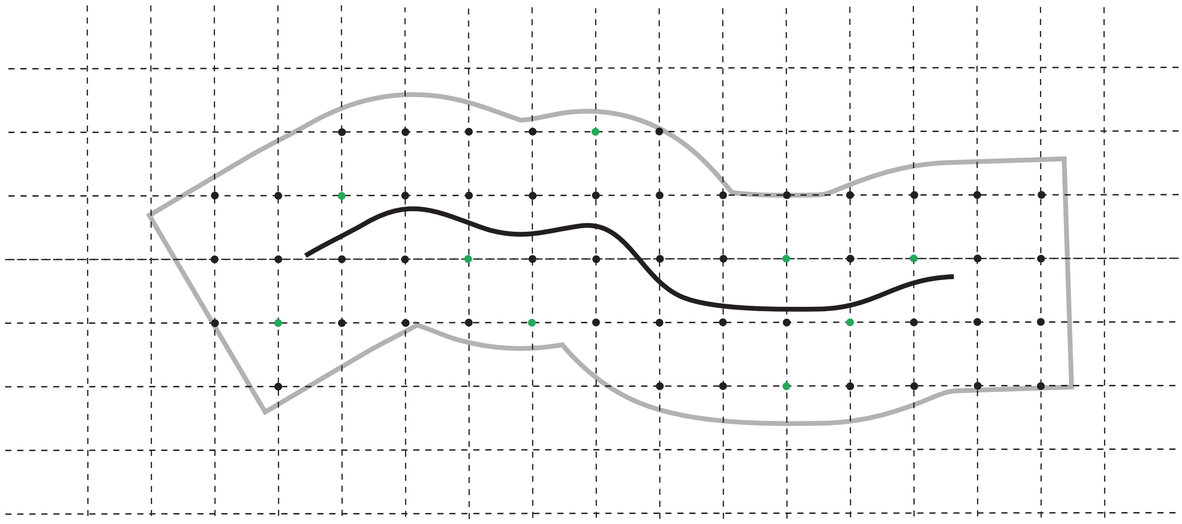
$$|\nabla\Phi_{v_1,v_2'}(v_1')| = |\nabla\varphi_2(v_1' + v_2' - v_1) - \nabla\varphi_1(v_1')| \geq c_1 > 0.$$

Set for $U_1 \subset W_1$,

$$\mathcal{N}(U_1) = \sup_{v_1 \in \tilde{V}_1, v_2' \in \tilde{V}_2} \#\{w_1' \in U_1 : v_1' \in \Pi_{v_1,v_2'}(C_1 R^{-\frac{1}{2}})\},$$

where $A(r)$ denotes the $r-$neighborhood of the set $A$. In other words, $\mathcal{N}(U_1)$ is the maximum number of green points in all possible pictures like the next one. The constant $C_1$ will be determined below.

Figure 5. The curve represents the set $\Pi_{v_1,v_2'}$ and the region around it is its $C_1 R^{-\frac{1}{2}}$ neighborhood. In black we plotted the points of $\mathcal{V}_1$ that are in this neighborhood, and for some $U_1 \subset W_1$, the points in green are the first components of the points of $U_1$ that are in this neighborhood.



**Lemma 9.1.** *For $P \in \mathcal{P}$ and $U_j \subset W_j(P)$, $j = 1, 2$,*

$$\left\| \sum_{w_1 \in U_1, w_2 \in U_2} p_{w_1} p_{w_2} \right\|_2^2 \lesssim R^{c\delta - \frac{(n-2)}{2}} \mathcal{N}(U_1) \# U_1 \# U_2. \tag{17}$$

*Proof.* Recall first that $p_{w_1}, p_{w_2} \in L^2(\mathbb{R}^n)$ by Lemma 6.1 $(i)$. We write

$$\left\| \sum_{w_1 \in U_1, w_2 \in U_2} p_{w_1} p_{w_2} \right\|_2^2 = \sum_{w_1, w_1' \in U_1, w_2, w_2' \in U_2} I_{w_1, w_1', w_2'},$$

where

$$I_{w_1,w_1',w_2'} = \sum_{w_2 \in W_2} \int p_{w_1} p_{w_2} \overline{p_{w_1'} p_{w_2'}}.$$

Now,

$$\int p_{w_1} p_{w_2} \overline{p_{w_1'} p_{w_2'}} = \int \widehat{p_{w_1} p_{w_2}} \overline{\widehat{p_{w_1'} p_{w_2'}}} = \int (\widehat{p_{w_1}} * \widehat{p_{w_2}}) \overline{(\widehat{p_{w_1'}} * \widehat{p_{w_2'}})}.$$

By Lemma 6.1 $(e)$, the $\widehat{p_{w_j}}$ are measures for which, with $C_2 = 2n(1 + C_0)$,

$$\operatorname{spt}(\widehat{p_{w_1}} * \widehat{p_{w_2}}) \subset B((v_1 + v_2, \varphi_1(v_1) + \varphi_2(v_2)), 2C_2 R^{-\frac{1}{2}}),$$
$$\operatorname{spt}(\widehat{p_{w_1'}} * \widehat{p_{w_2'}}) \subset B((v_1' + v_2', \varphi_1(v_1') + \varphi_2(v_2')), 2C_2 R^{-\frac{1}{2}}).$$

Hence $\int p_{w_1}p_{w_2}\overline{p_{w_1'}p_{w_2'}}=0$ unless

$$|v_1+v_2-(v_1'+v_2')|\leq 4C_2R^{-\frac{1}{2}}$$

and

$$|\varphi_1(v_1)+\varphi_2(v_2)-(\varphi_1(v_1')+\varphi_2(v_2'))|\leq 4C_2R^{-\frac{1}{2}}.$$

If $I_{w_1,w_1',w_2'}\neq 0$, there is $v_2$ such that the two inequalities above hold. Thus,

$$\begin{aligned}|\Phi_{v_1,v_2'}(v_1')| &= |\varphi_1(v_1)+\varphi_2(v_1'+v_2'-v_1)-(\varphi_1(v_1')+\varphi_2(v_2'))|\\ &\leq |\varphi_1(v_1)+\varphi_2(v_2)-(\varphi_1(v_1')+\varphi_2(v_2'))|+|\varphi_2(v_2)-\varphi_2(v_1'+v_2'-v_1)|\\ &\leq (4C_2+4C_2\|\nabla\varphi_2\|_\infty)R^{-\frac{1}{2}}\\ &\leq 8C_2^2R^{-\frac{1}{2}}.\end{aligned}$$

**Claim 9.2.** *$v_1'$ is contained in $\Pi_{v_1,v_2'}(C_1R^{-\frac{1}{2}})$ where $C_1=\frac{8C_2^2}{c_1}$.*

To prove the claim, suppose w.l.o.g. that $\Phi_{v_1,v_2'}(v_1')>0$ and that there is no point of $\Pi_{v_1,v_2'}$ in $B\left(v_1',\frac{8C_2^2}{c_1}R^{-\frac{1}{2}}\right)$. Since $|\nabla\Phi_{v_1,v_2'}|>0$, the minimum value of $\Phi$ in $B(v_1',\frac{8C_2^2}{c_1}R^{-\frac{1}{2}})$ occurs at some $b$ with $|v_1'-b|=\frac{8C_2^2}{c_1}R^{-\frac{1}{2}}$, and $\Phi_{v_1,v_2'}(b)>0$ by the intermediate value theorem. This way for some $c\in[v_1',b]$:

$$\begin{aligned}8C_2^2R^{-\frac{1}{2}}-\Phi_{v_1,v_2'}(b) &\geq \Phi_{v_1,v_2'}(v_1')-\Phi_{v_1,v_2'}(b)\\ &= |\Phi_{v_1,v_2'}(v_1')-\Phi_{v_1,v_2'}(b)|\\ &= |\nabla\Phi_{v_1,v_2'}(c)||v_1'-b|\\ &\geq c_1\frac{8C_2^2}{c_1}R^{-\frac{1}{2}},\end{aligned}$$

which implies $\Phi_{v_1,v_2'}(b)=0$ and this is a contradiction. Use this constant $C_1$ to define $\mathcal{N}(U_1)$ above. Hence the left hand side of (17) is

$$\sum_{w_1\in U_1}\sum_{w_2'\in U_2}\sum_{\substack{w_1'\in U_1\\ v_1'\in\Pi_{v_1,v_2'}(C_1R^{-\frac{1}{2}})}}\sum_{\substack{w_2\in U_2\\ v_2\in B(v_1'+v_2'-v_1,4C_2R^{-\frac{1}{2}})}} p_{w_1}p_{w_2}\overline{p_{w_1'}p_{w_2'}}.$$

Given $w_1,w_2',w_1'$, there are boundedly many points $v_2$ in the above sum. Since all the tubes $T_{w_2}$ meet $R^\delta P$, there are at most $O(R^{c\delta})$ points $w_2$ if $v_2$ is fixed because $y_2\in\mathcal{Y}$ are $\sqrt{R}$–separated. **By transversality** between the tubes $T_{w_1}$ and $T_{w_2}$, the measure of their intersection is $\lesssim R^{\frac{n}{2}}$. By parts $(g)$ and $(c)$ of Lemma 6.1 the product $p_{w_1}p_{w_2}\overline{p_{w_1'}p_{w_2'}}$ decays very fast off this intersection and it is uniformly $\lesssim R^{1-n}$. These give

$$\left|\int p_{w_1}p_{w_2}\overline{p_{w_1'}p_{w_2'}}\right|\lesssim R^{-\frac{(n-2)}{2}}.$$

Therefore for fixed $w_1,w_2',w_1'$,

$$\sum_{\substack{w_2\in U_2\\ v_2\in B(v_1'+v_2'-v_1,4C_2R^{-\frac{1}{2}})}} p_{w_1}p_{w_2}\overline{p_{w_1'}p_{w_2'}}\lesssim R^{c\delta-\frac{(n-2)}{2}}.$$

The lemma follows from this. □

The proof of the theorem will be finished by the following lemma.

**Lemma 9.3.** *For any dyadic integers $1 \le \kappa_1, \kappa_2, \lambda \le R^{2n}$, $Q \in \mathcal{Q}$, $P \in \mathcal{Q}(\kappa_1, \kappa_2)$, $P \subset 2Q$,*

$$\mathcal{N}(W_1^{\not\sim Q}(P, \lambda, \kappa_1, \kappa_2)) \lesssim R^{c\delta} \frac{\#W_2}{\lambda \kappa_2}.$$

Let us first see why this lemma implies (14). For any $P \in \mathcal{Q}(\kappa_1, \kappa_2)$, $\#U_2(P) \le \#W_2(P) \le 2\kappa_2$. Using this, the previous lemma and the definitions we gave throughout these notes,

$$\begin{aligned}
&\sum_{P \in Q(\kappa_1,\kappa_2), P \subset 2Q} \left\| \sum_{w_1 \in W_1^{\not\sim Q}(P,\lambda,\kappa_1,\kappa_2)} \sum_{w_2 \in U_2(P)} p_{w_1} p_{w_2} \right\|_{L^2(P)}^2 \\
&\lesssim R^{c\delta - \frac{(n-2)}{2}} \sum_{P \in Q(\kappa_1,\kappa_2), P \subset 2Q} \mathcal{N}(W_1^{\not\sim Q}(P, \lambda, \kappa_1, \kappa_2)) \#W_1^{\not\sim Q}(P, \lambda, \kappa_1, \kappa_2) \#U_2(P) \\
&\lesssim R^{2c\delta - \frac{(n-2)}{2}} \frac{\#W_2}{\lambda \kappa_2} \sum_{P \in Q(\kappa_1,\kappa_2), P \subset 2Q} \#W_1^{\not\sim Q}(P, \lambda, \kappa_1, \kappa_2) \#U_2(P) \\
&\le R^{2c\delta - \frac{(n-2)}{2}} \frac{2\#W_2}{\lambda} \sum_{P \in Q(\kappa_1,\kappa_2), P \subset 2Q} \#W_1^{\not\sim Q}(P, \lambda, \kappa_1, \kappa_2) \\
&\le R^{2c\delta - \frac{(n-2)}{2}} \frac{2\#W_2}{\lambda} \sum_{w_1 \in W_1(\lambda,\kappa_1,\kappa_2)} \#\{P \in \mathcal{Q}(\kappa_1, \kappa_2) : T_{w_1} \cap R^\delta P \neq \emptyset\} \\
&\le 4R^{2c\delta - \frac{(n-2)}{2}} \#W_1 \#W_2.
\end{aligned}$$

*Proof of Lemma 9.3.* We need to show that for any $v_1 \in \tilde{V}_1, v_2' \in \tilde{V}_2$ and $P_0 \in \mathcal{Q}(\kappa_1, \kappa_2)$, $P_0 \subset 2Q$,

$$\#\{w_1' \in W_1^{\not\sim Q}(P_0, \lambda, \kappa_1, \kappa_2) : v_1' \in \Pi_{v_1, v_2'}(C_1 R^{-\frac{1}{2}})\} \lesssim R^{c\delta} \frac{\#W_2}{\lambda \kappa_2}.$$

Set

$$W_1^{\not\sim Q}(\Pi_{v_1,v_2'}) := \{w_1' \in W_1^{\not\sim Q}(P_0, \lambda, \kappa_1, \kappa_2) : v_1' \in \Pi_{v_1,v_2'}(C_1 R^{-\frac{1}{2}})\}.$$

Let $w_1' \in W_1^{\not\sim Q}(\Pi_{v_1,v_2'})$. Then by definition $T_{w_1'} \cap R^\delta P_0 \neq \emptyset$ and $Q \cap 10Q(w_1', \lambda, \kappa_1, \kappa_2) = \emptyset$. Since $P_0 \subset 2Q$,

$$d(P_0, 2Q(w_1', \lambda, \kappa_1, \kappa_2)) \ge R^{1-\delta},$$

so by (11)

$$\#\{P \in \mathcal{Q}(\kappa_1, \kappa_2) : T_{w_1'} \cap R^\delta P \neq \emptyset, d(P, P_0) \ge R^{1-\delta}\} \ge \lambda R^{-n\delta},$$

because

$$\{P \in \mathcal{Q}(\kappa_1, \kappa_2) : T_{w_j} \cap R^\delta P \neq \emptyset, P \cap Q(w_j, \lambda, \kappa_1, \kappa_2) \neq \emptyset\} \subset \{P \in \mathcal{Q}(\kappa_1, \kappa_2) : T_{w_1'} \cap R^\delta P \neq \emptyset, d(P, P_0) \ge R^{1-\delta}\}.$$

Since $\kappa_2 \le \#W_2(P) \le 2\kappa_2$ for $P \in \mathcal{Q}(\kappa_1, \kappa_2)$, we get

(18)
$$\begin{aligned}
\#\{(P, w_1', w_2) \in \mathcal{Q}(\kappa_1, \kappa_2) \times W_1^{\not\sim Q}(\Pi_{v_1,v_2'}) \times W_2 : T_{w_1'} \cap R^\delta P \neq \emptyset, T_{w_2} \cap R^\delta P \neq \emptyset, d(P, P_0) \ge R^{1-\delta}\} \\
\gtrsim \lambda R^{-n\delta} \#W_1^{\not\sim Q}(\Pi_{v_1,v_2'}) \kappa_2.
\end{aligned}$$

We shall prove Lemma 9.3 by finding an upper bound for the left hand side of this inequality. This is accomplished by

**Lemma 9.4.** *Let $w_2 \in W_2$ and set*

$$S = \{(P, w_1') \in \mathcal{Q}(\kappa_1, \kappa_2) \times W_1^{\not\sim Q}(\Pi_{v_1,v_2'}) : T_{w_1'} \cap R^\delta P \neq \emptyset, T_{w_2} \cap R^\delta P \neq \emptyset, d(P, P_0) \geq R^{1-\delta}\}$$

*(observe that this is a slice of the set we dealt with previously). Then $\#S \lesssim R^{c\delta}$.*

This lemma finishes the proof since the left hand side of (18) is $\leq \#S \cdot \#W_2 \lesssim R^{c\delta} \#W_2$. Hence,

$$\lambda R^{-n\delta} \#W_1^{\not\sim Q}(\Pi_{v_1,v_2'})\kappa_2 \lesssim R^{c\delta} \#W_2 \iff \#W_1^{\not\sim Q}(\Pi_{v_1,v_2'}) \lesssim R^{(c+n)\delta} \frac{\#W_2}{\lambda \kappa_2}.$$

$\square$

*Proof of Lemma 9.4.* Define the conical set

$$C_{v_1,v_2'} = \{s(u,1) \in \mathbb{R}^{n-1} \times \mathbb{R} : u \in \nabla\varphi_1(\Pi_{v_1,v_2'}), |s| < 2R\}.$$

For $w_1' \in W_1^{\not\sim Q}(\Pi_{v_1,v_2'})$, we have $v_1' \in \Pi_{v_1,v_2'}(C_1 R^{-\frac{1}{2}})$ and $T_{w_1'} \cap R^\delta P_0 \neq \emptyset$.

**Claim 9.5.**

$$\bigcup_{w_1' \in W_1^{\not\sim Q}(\Pi_{v_1,v_2'})} T_{w_1'} \subset C_{v_1,v_2'}(C_3 R^{\frac{1}{2}+\delta}) + P_0,$$

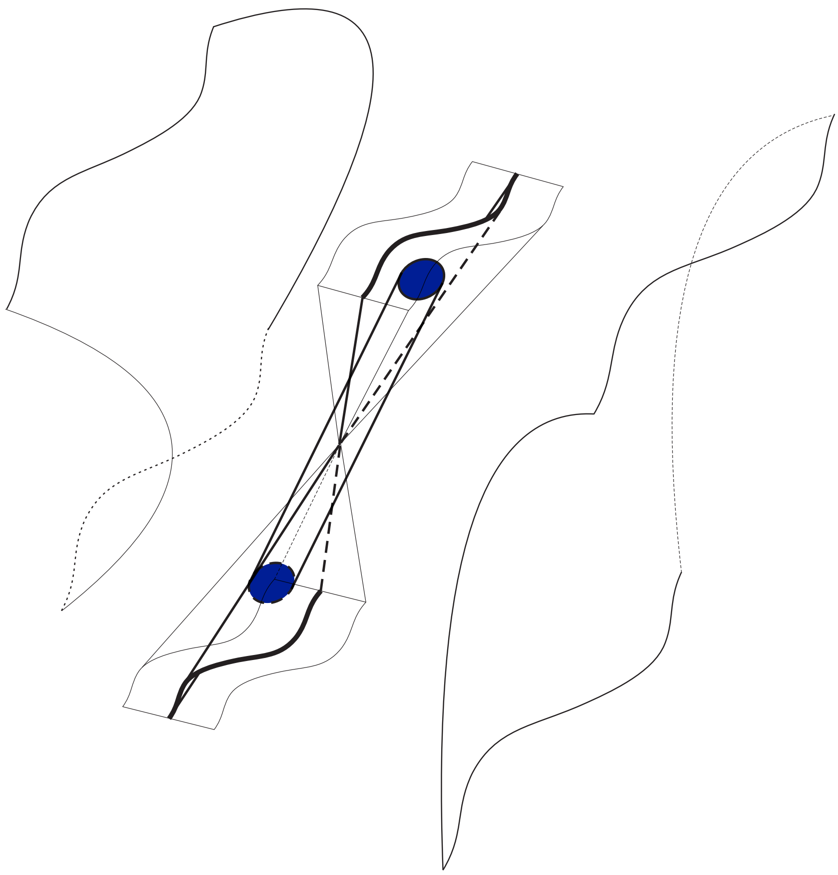
*for some constant $C_3 \geq 1$.*

Indeed, let $y \in T_{w_1'} \cap R^\delta P_0$ and consider $T_{w_1'} - y$. This is a tube that crosses the origin, but also has orientation $(\nabla\varphi_1(v_1'), 1)$. Since $v_1' \in \Pi_{v_1,v_2'}(C_1 R^{-\frac{1}{2}})$, there is $\tilde{v}_1' \in \Pi_{v_1,v_2'}$ such that $|v_1' - \tilde{v}_1'| \leq C_1 R^{-\frac{1}{2}}$, hence

$$|s(\nabla\varphi_1(v_1')) - s(\nabla\varphi_1(\tilde{v}_1'))| \leq \|\nabla\varphi_1\|_\infty R C_1 R^{-\frac{1}{2}} \lesssim_{\varphi_1} R^{\frac{1}{2}}.$$

Since $T_{w_1'} - y$ has radius $R^{\frac{1}{2}}$, it follows that we can fit it in a $C_3 R^{\frac{1}{2}+\delta}$ neighborhood of $C_{v_1,v_2'}$ and the claim follows.

Figure 6. The surface with "thick" boundary is the conical set $C_{v_1,v_2'}$. Here we represented a $C^2R^{\frac{1}{2}+\delta}-$neighborhood of it, and also the tube $T_{w_1'}-y$ that is contained in it.



**Claim 9.6.** *If* $(P, w_1') \in S$*, then*

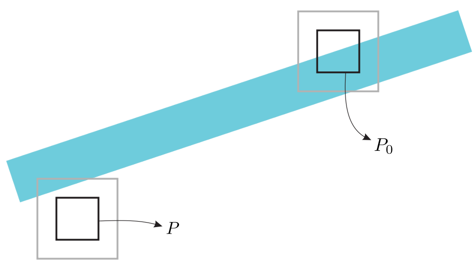
$$P \subset C_{v_1,v_2'}(C_4R^{\frac{1}{2}+\delta}) + P_0.$$

Indeed, in particular we have $w_1' \in W_1^{\not\sim Q}(\Pi_{v_1,v_2'})$, so $T_{w_1'} \subset C_{v_1,v_2'}(C_3R^{\frac{1}{2}+\delta}) + P_0$ by the previous lemma. We also have $T_{w_1'} \cap R^\delta P \neq \emptyset$. Let $\tilde{T}_{w_1'}$ be a "dilated" version of $T_{w_1'}$ with length $\gtrsim R$ and radius $\gtrsim R^{\frac{1}{2}}$ just so that $P \subset \tilde{T}_{w_1'}$. Then we can find a constant $C_4$ for which

$$P \subset \tilde{T}_{w_1'} \subset C_{v_1,v_2'}(C_4R^{\frac{1}{2}+\delta}) + P_0$$

and this claim follows.

Figure 7. Tube $T_{w_1'}$ intersecting both $R^\delta P$ and $R^\delta P_0$.



**Claim 9.7.** *If $(P, w_1') \in S$, then*

$$P \subset C_{v_1,v_2'}(R^{\frac{1}{2}+\delta}, R^{1-\delta}, R, P_0),$$

*where for a suitable constant $c_2 > 0$,*

$$C_{v_1,v_2'}(R^{\frac{1}{2}+\delta}, R^{1-\delta}, R, P_0) = C_{v_1,v_2'}(C_4 R^{\frac{1}{2}+\delta}) \cap \{(x,t) : c_2 R^{1-\delta} \leq |t| \leq R\} + P_0.$$

This follows from $d(P, P_0) \geq R^{1-\delta}$ and the previous claim.
Furthermore, $T_{w_2} \cap R^\delta P \neq \emptyset$ if $(P, w_1') \in S$, so

$$\bigcup_{(P,w_1') \in S \text{ for some } w_1'} P \subset R^\delta T_{w_2} \cap C_{v_1,v_2'}(R^{\frac{1}{2}+\delta}, R^{1-\delta}, R, P_0),$$

where

$$R^\delta T_{w_2} = \{(x,t) : |t| \leq R, |x - (y_2 - t\nabla\varphi_2(v_2))| \leq (2 + C_0) R^{\frac{1}{2}+\delta}\},$$

with $C_0$ as in Section 2. We claim that

$$R^\delta T_{w_2} \cap C_{v_1,v_2'}(R^{\frac{1}{2}+\delta}, R^{1-\delta}, R, P_0) \subset B(y_0, R^{\frac{1}{2}+c\delta}) \tag{19}$$

for some $y \in \mathbb{R}^n$ and some positive constant $c$. This is a consequence of the fact that the tube $T_{w_2}$ intersects transversally the surface $C_{v_1,v_2'}$ due to our basic assumptions on the functions $\varphi_j$, and we will prove it in the next Lemma. From (19) it follows that for each $w_1'$ there are $O(R^{c\delta})$ cubes $P$ with $(P, w_1') \in S$. Since $d(P, P_0) \geq R^{1-\delta}$ the number of possible $w_1'$ for which $T_{w_1'}$ meets both $R^\delta P$ and $R^\delta P_0$ is also $O(R^{c\delta})$. Lemma 9.4 follows from this. ☐

For a smooth hypersurface $S \subset \mathbb{R}^n$ we denote by $\mathrm{Tan}(S, p)$ the tangent space of $S$ at $p$ considered as an $(n-1)-$dimensional linear subspace of $\mathbb{R}^n$. Then the geometric tangent space is $\mathrm{Tan}(S, p) + p$.

**Lemma 9.8.** *Let $c > 0$ and let $\Pi$ be a smooth hypersurface in $\mathbb{R}^{n-1}$ with $\Pi \subset B(0, 1)$ such that $\Pi = \{v \in \mathbb{R}^{n-1} : \Phi(v) = 0\}$ where $\Phi$ is of class $C^2$ and $|\nabla\Phi(v)| \geq c$ for all $v \in B(0, 1)$. Set*

$$C(\Pi) = \{s(x, 1) \in \mathbb{R}^{n-1} \times \mathbb{R} : 0 \leq s \leq 1, x \in \Pi\}.$$

*For any $y, v \in \mathbb{R}^{n-1}$, $v \neq 0$, let $l_{y,v}$ be the line in the direction $(v, 1)$ through $(y, 0)$, that is,*

$$l_{y,v} = \{(x,t) \in \mathbb{R}^{n-1} \times \mathbb{R} : x = y + vt\}.$$

*Suppose for some $v \in B(0, 1)$,*

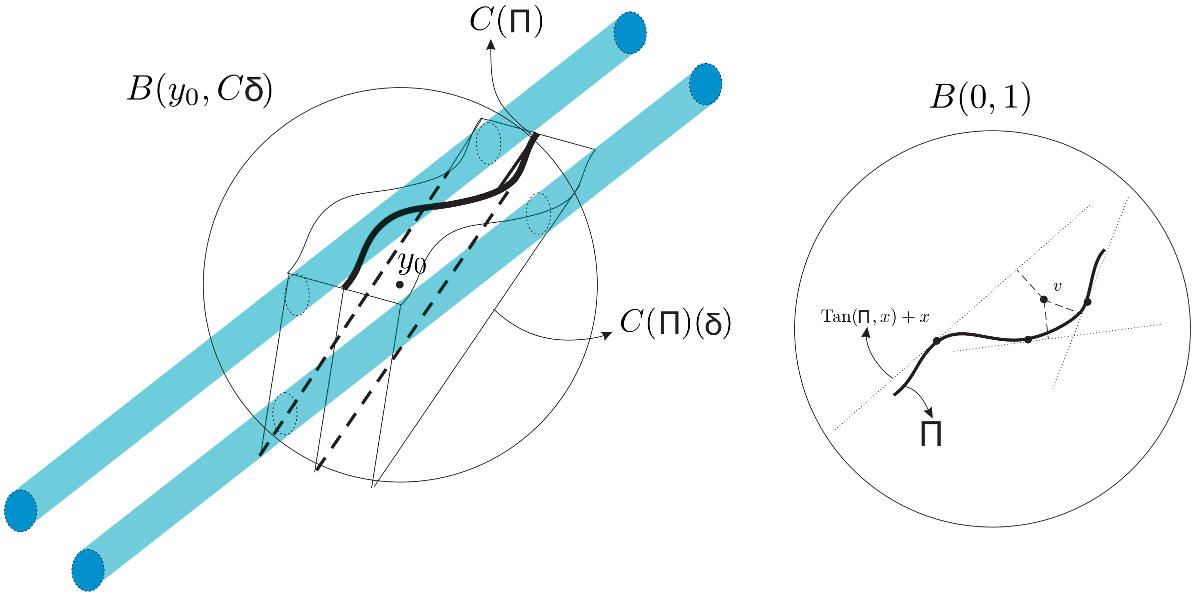
$$d(v, \mathrm{Tan}(\Pi, x) + x) \geq c \text{ for all } x \in \Pi.$$

*Then for any $y \in \mathbb{R}^{n-1}$ and $0 < \delta < 1$,*

$$l_{y,v}(\delta) \cap C(\Pi)(\delta) \subset B(y_0, C\delta) \tag{20}$$

*for some $y_0 \in \mathbb{R}^n$, where $C$ depends only on $c$ and $n$.*

Figure 8. Pictorial description of Lemma 9.8.




*Proof.* We claim that for all $p \in C(\Pi)$,

$$d((v,1), \mathrm{Tan}(C(\Pi), p)) \geq \frac{c}{2}.$$

This means that $l_{y,v}$ meets transversally $C(\Pi)$ if it meets it at all. This gives (20) and proves the lemma. To check this inequality, let $p = s(x,1) \in C(\Pi)$, $x \in \Pi$. Note that

$$\mathrm{Tan}(C(\Pi), p) = \mathrm{Tan}(\Pi, x) \times \{0\} + \{t(x,1) : t \in \mathbb{R}\}.$$

Suppose $d((v,1), \mathrm{Tan}(C(\Pi), p)) < \frac{c}{2}$. Then there are $u \in \mathrm{Tan}(\Pi, x)$ and $t \in \mathbb{R}$ such that $|(v,1) - (u + tx, t)| < \frac{c}{2}$. This gives $|v - u - tx| < \frac{c}{2}$ and $|1 - t| < \frac{c}{2}$. Thus $|v - (u + x)| < c$ (because $|x| < 1$) and so $d(v, \mathrm{Tan}(\Pi, x) + x) < c$ giving a contradiction. This completes the proof of the lemma. □

Let us finish the proof by showing that Lemma 9.8 implies (19). Recall that the maps $\nabla\varphi_j$, $j = 1, 2$, are diffeomorphisms. Define

$$\Psi(v) = \varphi_1(v_1) + \varphi_2((\nabla\varphi_1)^{-1}(v) + v_2' - v_1) - \varphi_1((\nabla\varphi_1)^{-1}(v)) - \varphi_2(v_2')$$

when $v \in \nabla\varphi_1(V_1)$. Then

$$\nabla\varphi_1(\Pi_{v_1,v_2'}) \subset \{v \in \mathbb{R}^{n-1} : \Psi(v) = 0\}.$$

By a straightforward computation,

$$\nabla\Psi(\nabla\varphi_1(v_1')) = D(\nabla\varphi_1)(v_1')^{-1}(\nabla\varphi_2(v_1' + v_2' - v_1) - \nabla\varphi_1(v_1')).$$

The normal vector to the surface $\nabla\varphi_1(\Pi_{v_1,v_2'})$ at $\nabla\varphi_1(v_1')$ is parallel to this gradient, so the tangent space is

$$\mathrm{Tan}(\nabla\varphi_1(\Pi_{v_1,v_2'}), \nabla\varphi_1(v_1')) = \{x : x \cdot \nabla\Psi(v_1') = 0\}.$$

Let $w_2 = (y_2, v_2) \in W_2$. Using the inequalities in the setting of Section 2

$$
\begin{aligned}
&|\nabla\Psi(\nabla\varphi_1(v_1'))\cdot(\nabla\varphi_2(v_2)-\nabla\varphi_1(v_1'))|\\
&\quad= |D(\nabla\varphi_1)(v_1')^{-1}(\nabla\varphi_2(v_2)-\nabla\varphi_1(v_1'))\cdot(\nabla\varphi_2(v_2)-\nabla\varphi_1(v_1'))|\\
&\quad- |D(\nabla\varphi_1)(v_1')^{-1}(\nabla\varphi_2(v_1'+v_2'-v_1)-\nabla\varphi_2(v_2))\cdot(\nabla\varphi_2(v_2)-\nabla\varphi_1(v_1'))|\\
&\quad\geq \frac{c_0}{2}.
\end{aligned}
$$

Then

$$
d(\nabla\varphi_2(v_2), \mathrm{Tan}(\nabla\varphi_1(\Pi_{v_1,v_2'}), \nabla\varphi_1(v_1')) + \nabla\varphi_1(v_1')) \approx |\nabla\Psi(\nabla\varphi_1(v_1'))\cdot(\nabla\varphi_2(v_2)-\nabla\varphi_1(v_1'))| \geq \frac{c_0}{2}.
$$

We now apply Lemma 9.8 to the surface $\Pi = \nabla\varphi_1(\Pi_{v_1,v_2'})$ with $v = \nabla\varphi_2(v_2)$ and $\delta$ replaced by $R^{\delta-\frac{1}{2}}$. Scaling by $R$ (19) follows and the theorem is proven.

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