

NOTES ON THE KAKEYA MAXIMAL CONJECTURE AND RELATED PROBLEMS

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ABSTRACT. We discuss the relation between the restriction conjecture, the Kakeya maximal conjecture and the Kakeya conjecture. We follow [4] and [5] very closely.

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1. INITIAL CONCEPTS

Definition 1.1 (Hausdorff measure). Let $A \subset \mathbb{R}^n$, $0 \leq s < \infty$, $0 < \delta \leq \infty$. We write

$$\mathcal{H}_\delta^s(A) := \inf \left\{ \sum_{j=1}^{\infty} \alpha(s) \left(\frac{\text{diam } C_j}{2} \right)^s : A \subset \bigcup_{j=1}^{\infty} C_j, \text{diam } C_j \leq \delta \right\},$$

where

$$\alpha(s) := \frac{\pi^{\frac{s}{2}}}{\Gamma(\frac{s}{2} + 1)}.$$

Define also:

$$\mathcal{H}^s(A) := \lim_{\delta \rightarrow 0} \mathcal{H}_\delta^s(A) = \sup_{\delta > 0} \mathcal{H}_\delta^s(A).$$

We call \mathcal{H}^s the *s-dimensional Hausdorff measure on \mathbb{R}^n* .

Definition 1.2. The *Hausdorff dimension* of $A \subset \mathbb{R}^n$ is

$$\dim A := \inf \{s : \mathcal{H}^s(A) = 0\} = \sup \{s : \mathcal{H}^s(A) = \infty\}.$$

We also have the following characterization of the Hausdorff dimension:

$$\dim A = \inf \left\{ s : \forall \varepsilon > 0, \exists E_1, E_2, \dots \subset \mathbb{R}^n : A \subset \bigcup_j E_j \text{ and } \sum_j (\text{diam } E_j)^s < \varepsilon \right\}.$$

A detailed exposition of the properties of the Hausdorff measure can be found in [1]. For $A \subset \mathbb{R}^n$ define

$$A_\delta = \{x \in \mathbb{R}^n : d(x, A) < \delta\}.$$

Definition 1.3. The *lower Minkowski dimension* of a bounded set $A \subset \mathbb{R}^n$ is

$$\underline{\dim}_M A := \inf\{s : \liminf_{\delta \rightarrow 0} \delta^{s-n} |A_\delta| = 0\},$$

and the *upper Minkowski dimension* of A is

$$\overline{\dim}_M A := \inf\{s : \limsup_{\delta \rightarrow 0} \delta^{s-n} |A_\delta| = 0\}.$$

More about Minkowski dimension can be found in [3]. We conclude this section with the following conjecture, which is one of the main objects of study here.

Conjecture 1.4 (Kakeya conjecture). *Every Besicovitch set in \mathbb{R}^n has Hausdorff dimension n .*

2. KAKEYA MAXIMAL FUNCTION

Let us start with the following definition:

Definition 2.1. A Borel set B in \mathbb{R}^n , $n \geq 2$, is a *Besicovitch set* or *Kakeya set* if it has Lebesgue measure zero and for every $e \in \mathbb{S}^{n-1}$ there is $b \in \mathbb{R}^n$ such that $\{te + b : 0 < t < 1\} \subset B$. In other words, B contains a line segment of unit length in every direction.

The existence of Besicovitch sets (even compact ones) is proved in [4] on page 143.

Definition 2.2. For $a \in \mathbb{R}^n$, $e \in \mathbb{S}^{n-1}$ and $\delta > 0$, define the *tube* $T_e^\delta(a)$ by

$$T_e^\delta(a) = \{x \in \mathbb{R}^n : |(x - a) \cdot e| \leq 1/2, |(x - a) - ((x - a) \cdot e)e| \leq \delta\}.$$

Observe that $|T_e^\delta(a)| = \alpha(n-1)\delta^{n-1}$, where $\alpha(n-1)$ is the Lebesgue measure of the unit ball in \mathbb{R}^{n-1} .

Definition 2.3. The *Kakeya maximal function* with width δ of $f \in L_{loc}^1(\mathbb{R}^n)$ is the function $\mathcal{K}_\delta f : \mathbb{S}^{n-1} \rightarrow [0, \infty]$ given by

$$\mathcal{K}_\delta f(e) = \sup_{a \in \mathbb{R}^n} \frac{1}{|T_e^\delta(a)|} \int_{T_e^\delta(a)} |f(x)| dx.$$

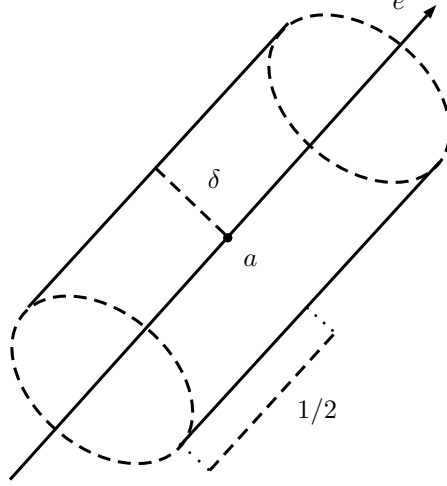
The following proposition is immediate by the definition given above:

Proposition 2.4. *For all $0 < \delta < 1$ and $f \in L_{loc}^1(\mathbb{R}^n)$,*

$$(1) \quad \|\mathcal{K}_\delta f\|_{L^\infty(\mathbb{S}^{n-1})} \leq \|f\|_{L^\infty(\mathbb{R}^n)} \quad \text{and}$$

$$(2) \quad \|\mathcal{K}_\delta f\|_{L^\infty(\mathbb{S}^{n-1})} \leq \alpha(n-1)^{-1} \delta^{1-n} \|f\|_{L^1(\mathbb{R}^n)}.$$

Figure 1. Tube.



Remark 2.5. It is natural to look for inequalities like

$$(3) \quad \|\mathcal{K}_\delta f\|_{L^q(\mathbb{S}^{n-1})} \leq C \|f\|_{L^p(\mathbb{R}^n)},$$

where C does not depend on δ , $p < \infty$ and $f \in L^p(\mathbb{R}^n)$. Let us use the existence of Besicovitch sets to prove that such estimates are not possible. Let $B \subset \mathbb{R}^n$ be a compact Besicovitch set and let

$$B_\delta = \{x \in \mathbb{R}^n : d(x, B) < \delta\}, \quad f = \chi_{B_\delta}.$$

It is clear that $\mathcal{K}_\delta f \leq 1$. For each $e \in \mathbb{S}^{n-1}$ let a be the midpoint of some line segment of length 1 on the direction of e . This way, $T_e^\delta(a) \subset B_\delta$ and we have

$$\mathcal{K}_\delta f(e) \geq \frac{1}{|T_e^\delta(a)|} \int_{T_e^\delta(a)} |f(x)| dx = 1,$$

so $\mathcal{K}_\delta f = 1$ and $\|\mathcal{K}_\delta f\|_{L^q(\mathbb{S}^{n-1})} = |\mathbb{S}^{n-1}|^{\frac{1}{q}}$. On the other hand, $\|f\|_{L^p(\mathbb{R}^n)} = |B_\delta|^{\frac{1}{p}} \rightarrow 0$ as $\delta \rightarrow 0$, so there are no inequalities like (3).

Remark 2.6. A possible next step would be to prove estimates like

$$(4) \quad \|\mathcal{K}_\delta f\|_{L^p(\mathbb{S}^{n-1})} \leq C(p, n, \varepsilon) \delta^{-\varepsilon} \|f\|_{L^p(\mathbb{R}^n)},$$

for all $\varepsilon > 0$, $0 < \delta < 1$ and $f \in L^p(\mathbb{R}^n)$. However, this does not hold for $p < n$. Indeed, take $f = \chi_{B(0, \delta)}$. Since $B(0, \delta) \subset T_e^\delta(0)$ for all $e \in \mathbb{S}^{n-1}$, we have

$$\mathcal{K}_\delta f(e) = \frac{|B(0, \delta)|}{|T_e^\delta(0)|} = C \frac{\delta^n}{\delta^{n-1}} = C\delta.$$

On the other hand,

$$\|f\|_{L^p(\mathbb{R}^n)} = |B(0, \delta)|^{\frac{1}{p}} = C\delta^{\frac{n}{p}},$$

so (4) would become $\delta \leq C\delta^{n/p-\varepsilon}$, which is false for small δ if $p < n$ for some ε such that $n/p - \varepsilon > 1$. The *Keakeya maximal conjecture* wishes for the next best thing.

Conjecture 2.7 (Keakeya maximal conjecture). (4) holds if $p = n$, that is,

$$(5) \quad \|\mathcal{K}_\delta f\|_{L^n(\mathbb{S}^{n-1})} \leq C(n, \varepsilon) \delta^{-\varepsilon} \|f\|_{L^n(\mathbb{R}^n)},$$

for all $\varepsilon > 0$, $0 < \delta < 1$ and $f \in L^n(\mathbb{R}^n)$.

Remark 2.8. Interpolating (5) with (2) we get

$$\|\mathcal{K}_\delta f\|_{L^q(\mathbb{S}^{n-1})} \leq (C(n, \varepsilon) \delta^{-\varepsilon})^{1-\theta} (\delta^{1-n})^\theta \|f\|_{L^p(\mathbb{R}^n)},$$

for all $\varepsilon > 0$, $0 < \delta < 1$ and $f \in L^p(\mathbb{R}^n)$, where

$$\frac{1}{q} = \frac{1-\theta}{n} \quad \text{and} \quad \frac{1}{p} = \frac{1-\theta}{n} - \frac{\theta}{1}.$$

Solving this in θ , we get

$$\|\mathcal{K}_\delta f\|_{L^q(\mathbb{S}^{n-1})} \leq C(n, p, \varepsilon) \delta^{-(n/p-1+\varepsilon n(p-1)/p(n-1))} \|f\|_{L^p(\mathbb{R}^n)},$$

for all $1 \leq p \leq n$, $q = (n-1)p'$. Since $\varepsilon > 0$ is arbitrary and the constant C above already depends on p and n , this is the same as saying

$$\|\mathcal{K}_\delta f\|_{L^q(\mathbb{S}^{n-1})} \leq C(n, p, \varepsilon) \delta^{-(n/p-1+\varepsilon)} \|f\|_{L^p(\mathbb{R}^n)},$$

for all $\varepsilon > 0$, $0 < \delta < 1$, $f \in L^p(\mathbb{R}^n)$, $1 \leq p \leq n$ and $q = (n-1)p'$.

Definition 2.9. We say that $\{e_1, \dots, e_m\} \subset \mathbb{S}^{n-1}$ is a δ -separated subset of \mathbb{S}^{n-1} if $|e_j - e_k| \geq \delta$ for $j \neq k$. It is *maximal* if in addition for every $e \in \mathbb{S}^{n-1}$ there is some k for which $|e - e_k| < \delta$. We call T_1, \dots, T_m δ -separated δ -tubes if $T_k = T_{e_k}^\delta(a_k)$, $1 \leq k \leq m$, for some δ -separated subset $\{e_1, \dots, e_m\} \subset \mathbb{S}^{n-1}$ and some $a_1, \dots, a_m \in \mathbb{R}^n$.

Remark 2.10. In the definition above, $m \lesssim \delta^{1-n}$ for all δ -separated sets. Indeed, if $m > C\delta^{1-n-\varepsilon}$, by partially covering \mathbb{S}^{n-1} with m disjoint caps A_k of area $|A_k| \geq \tilde{C}\delta^{n-1}$ centered at e_k we would have:

$$|\mathbb{S}^{n-1}| \geq m|A_k| \geq C\delta^{1-n-\varepsilon} \tilde{C}\delta^{n-1} = C_0\delta^{-\varepsilon},$$

but the right-hand side goes to infinity as $\delta \rightarrow 0$, contradiction. By a similar argument, if the δ -separated set is maximal we have $m \approx \delta^{1-n}$.

Remark 2.11. If $e, e' \in \mathbb{S}^{n-1}$ and $|e - e'| \leq \delta$, then

$$(6) \quad \mathcal{K}_\delta f(e) \leq C(n) \mathcal{K}_\delta f(e').$$

Indeed, for $a \in \mathbb{R}^n$ arbitrary, fix $x \in T_e^\delta(a)$. Let b be the projection of x on the main axis of $T_e^\delta(a)$ and b' be the projection of b on the main axis of $T_{e'}^\delta(a)$. We have $\sin(\hat{b}\hat{a}b') < |e - e'| \leq \delta$ and then:

$$|b - b'| = |a - b| \sin(\hat{b}\hat{a}b') \lesssim \delta.$$

By the triangle inequality,

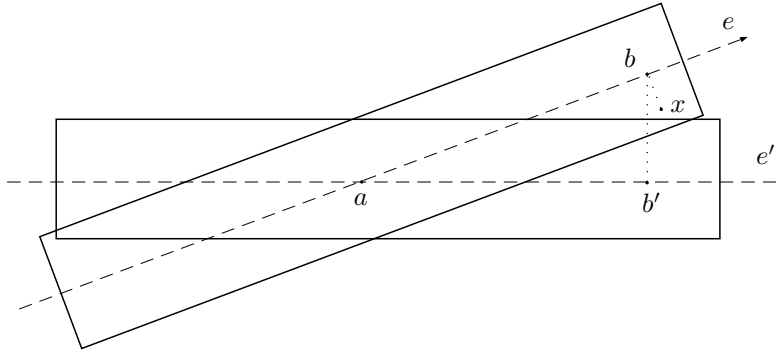
$$|x - b'| \leq |x - b| + |b - b'| \lesssim \delta,$$

so there is a constant M such that $x \in T_{e'}^{M\delta}(a)$ and we get $T_e^\delta(a) \subset T_{e'}^{M\delta}(a)$. Observe that we can cover $T_{e'}^{M\delta}(a)$ with $C(n) := 2^{n(\lceil \log C \rceil + 1)}$ (this is probably far from sharp) cylinders $T_{e'}^\delta(y_k)$, where $y_k \in \mathbb{R}^n$. Finally,

$$\begin{aligned} \frac{1}{|T_e^\delta(a)|} \int_{T_e^\delta(a)} |f(x)| dx &\leq \frac{1}{|T_{e'}^\delta(a)|} \int_{T_{e'}^{M\delta}(a)} |f(x)| dx \\ &\leq \sum_{k=1}^{C(n)} \frac{1}{|T_{e'}^\delta(y_k)|} \int_{T_{e'}^\delta(y_k)} |f(x)| dx \\ &\leq C(n) \mathcal{K}_\delta f(e'), \end{aligned}$$

and we get (6) by taking the supremum over a on the left-hand side.

Figure 2. Remark 2.11.



Proposition 2.12. *Let $1 < p < \infty$, $q = \frac{p}{p-1} = p'$, $0 < \delta < 1$ and $0 < M < \infty$. Suppose that*

$$\left\| \sum_{k=1}^m t_k \chi_{T_k} \right\|_{L^q(\mathbb{R}^n)} \leq M$$

whenever T_1, \dots, T_m are δ -separated δ -tubes and t_1, \dots, t_m are positive numbers with

$$\delta^{n-1} \sum_{k=1}^m t_k^q \leq 1.$$

Then

$$\|\mathcal{K}_\delta f\|_{L^p(\mathbb{S}^{n-1})} \leq C(n)M\|f\|_{L^p(\mathbb{R}^n)}.$$

Proof. Let $\{e_1, \dots, e_m\}$ be a maximal δ -separated subset of \mathbb{S}^{n-1} . Observe that the collection $\{\mathbb{S}^{n-1} \cap B(e_k, \delta)\}$ covers \mathbb{S}^{n-1} . If $e \in \mathbb{S}^{n-1} \cap B(e_k, \delta)$,

then $\mathcal{K}_\delta f(e) \leq C(n)\mathcal{K}_\delta f(e_k)$ by Remark 2.11. Hence

$$\begin{aligned}
\|K_\delta f\|_{L^p(\mathbb{S}^{n-1})}^p &\leq \sum_{k=1}^m \int_{\mathbb{S}^{n-1} \cap B(e_k, \delta)} |K_\delta f(e)|^p d\sigma(e) \\
&\leq \sum_{k=1}^m C(n)^p \int_{\mathbb{S}^{n-1} \cap B(e_k, \delta)} |K_\delta f(e_k)|^p d\sigma(e) \\
&\leq \sum_{k=1}^m C(n)^p |K_\delta f(e_k)|^p |\mathbb{S}^{n-1} \cap B(e_k, \delta)| \\
&\lesssim \sum_{k=1}^m (|K_\delta f(e_k)| \delta^{\frac{n-1}{p}})^p.
\end{aligned}$$

By the duality of l^p and l^q , for any $a_m \geq 0$, $k = 1, \dots, m$,

$$\left(\sum_{k=1}^m a_k^p \right)^{\frac{1}{p}} = \max \left\{ \sum_{k=1}^m a_k b_k; b_k \geq 0 \text{ and } \sum_{k=1}^m b_k^q = 1 \right\}.$$

This way,

$$\|K_\delta f\|_{L^p(\mathbb{S}^{n-1})} \lesssim \sum_{k=1}^m |K_\delta f(e_k)| \delta^{\frac{n-1}{p}} b_k = \delta^{n-1} \sum_{k=1}^m |K_\delta f(e_k)| t_k.$$

By choosing $t_k = \delta^{(1-n)/q} b_k$ we get $\delta^{n-1} \sum_{k=1}^m t_k^q = 1$. Let $\varepsilon > 0$ be small. There are points $a_k \in \mathbb{R}^n$ such that

$$|K_\delta f(e_k)| - \varepsilon \leq \frac{1}{T_{e_k}^\delta(a_k)} \int_{T_{e_k}^\delta(a_k)} |f(x)| dx,$$

hence

$$\begin{aligned}
\|K_\delta f\|_{L^p(\mathbb{S}^{n-1})} &\lesssim \delta^{n-1} \sum_{k=1}^m (|K_\delta f(e_k)| - \varepsilon + \varepsilon) t_k \\
&\leq \delta^{n-1} \sum_{k=1}^m t_k \frac{1}{T_{e_k}^\delta(a_k)} \int_{T_{e_k}^\delta(a_k)} |f(x)| dx + \varepsilon \\
&= \sum_{k=1}^m t_k \int_{T_{e_k}^\delta(a_k)} |f(x)| dx + \varepsilon \\
&= \int_{\mathbb{R}^n} \left(\sum_{k=1}^m t_k \chi_{T_{e_k}^\delta(a_k)} \right) |f(x)| dx + \varepsilon \\
&\leq \left\| \sum_{k=1}^m t_k \chi_{T_{e_k}^\delta(a_k)} \right\|_{L^q(\mathbb{R}^n)} \|f\|_{L^p(\mathbb{R}^n)} + \varepsilon \\
&\leq M \|f\|_{L^p(\mathbb{R}^n)} + \varepsilon.
\end{aligned}$$

By taking $\varepsilon \rightarrow 0$ we finish the proof. \square

Let us now prove a key lemma.

Lemma 2.13. *For any pair of directions $e_k, e_l \in \mathbb{S}^{n-1}$ and any pair of points $a, b \in \mathbb{R}^n$, we have the estimates*

$$(7) \quad \text{diam}(T_{e_k}^\delta(a) \cap T_{e_l}^\delta(b)) \lesssim \frac{\delta}{|e_k - e_l|}$$

and

$$(8) \quad |T_{e_k}^\delta(a) \cap T_{e_l}^\delta(b)| \lesssim \frac{\delta^n}{|e_k - e_l|}.$$

Proof. Pick $z \in T_{e_k}^\delta(a) \cap T_{e_l}^\delta(b)$. For any $y \in T_{e_k}^\delta(a)$ we have:

$$|(y - z) \cdot e_k| \leq |(y - a) \cdot e_k| + |(z - a) \cdot e_k| \leq 1.$$

Also, for any unit vector $v \perp e_k$:

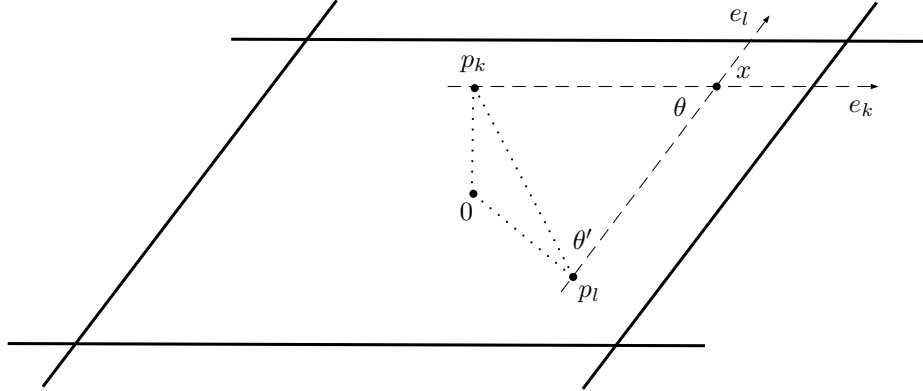
$$|(y - z) \cdot v| \leq |(y - a) \cdot v| + |(z - a) \cdot v| \leq 2\delta.$$

Then $z \in 2T_{e_k}^\delta(z)$. We conclude $z \in 2T_{e_l}^\delta(z)$ analogously, so

$$T_{e_k}^\delta(a) \cap T_{e_l}^\delta(b) \subset 2T_{e_k}^\delta(z) \cap 2T_{e_l}^\delta(z).$$

Combining this with the translation invariance of the Lebesgue measure, it suffices to prove (7) for $a = b = 0$. By switching the orientation of the tube we can assume $\theta \leq \pi/2$.

Figure 3. $T_{e_k}^\delta(0) \cap T_{e_l}^\delta(0)$.



Let $x \in T_{e_k}^\delta(0) \cap T_{e_l}^\delta(0)$ and let p_k and p_l be the projections of 0 on the lines $f(t) = x + te_k$ and $g(t) = x + te_l$, respectively. We can parametrize it by $x = p_k + t_k e_k$, where $|t_k| \leq 1/2$, so $|t_k| = |x - p_k|$. By the law of sines.

$$\frac{\sin(\theta)}{|p_k - p_l|} = \frac{\sin(\theta')}{|t_k|} \leq \frac{1}{|t_k|} \Rightarrow |t_k| \leq \frac{|p_k - p_l|}{\sin(\theta)} \leq \frac{2\delta}{\sin(\theta)}.$$

By basic plane geometry:

$$\frac{|e_k - e_l|}{2} = \sin\left(\frac{\theta}{2}\right) \leq \sin(\theta) \Rightarrow \sin(\theta) \gtrsim |e_k - e_l| \Rightarrow |t_k| \lesssim \frac{\delta}{|e_k - e_l|},$$

by what we just did. Finally, by the triangle inequality we have:

$$|x| \leq |p_k| + |t_k| \lesssim \delta + \frac{\delta}{|e_k - e_l|} \lesssim \frac{\delta}{|e_k - e_l|}.$$

Since x is arbitrary, this holds for every point in $T_{e_k}^\delta(0) \cap T_{e_l}^\delta(0)$. Finally,

$$\text{diam}(T_{e_k}^\delta(0) \cap T_{e_l}^\delta(0)) \leq \sup_{x \in T_{e_k}^\delta(0) \cap T_{e_l}^\delta(0)} 2|x| \lesssim \frac{\delta}{|e_k - e_l|},$$

which verifies the first part of this lemma. For the second part, observe that

$$T_{e_k}^\delta(0) \cap T_{e_l}^\delta(0) \subset \frac{C\delta}{|e_k - e_l|} T_{e_k}^{|e_k - e_l|}(0).$$

In fact, for any $x \in T_{e_k}^\delta(0) \cap T_{e_l}^\delta(0)$ we have $|x \cdot v| \leq \delta$ (for a unit vector $v \perp e_k$) and $x \in B(0, C\delta/|e_k - e_l|)$ for some C by what we just proved. This way,

$$|T_{e_k}^\delta(0) \cap T_{e_l}^\delta(0)| \leq \left| \frac{C\delta}{|e_k - e_l|} T_{e_k}^{|e_k - e_l|}(0) \right| \lesssim \frac{\delta}{|e_k - e_l|} \delta^{n-1} = \frac{\delta^n}{|e_k - e_l|}.$$

□

Now we will use the proposition above to prove the Kakeya maximal conjecture for $n = 2$.

Theorem 2.14. *For all $0 < \delta < 1$ and $f \in L^2(\mathbb{R}^2)$,*

$$\|\mathcal{K}_\delta f\|_{L^2(\mathbb{S}^1)} \leq C \sqrt{\log(1/\delta)} \|f\|_{\mathbb{R}^2},$$

with some absolute constant C .

Proof. Let $T_k = T_{e_k}^\delta(a_k)$, $k = 1, \dots, m$, be δ -separated δ -tubes and t_1, \dots, t_m positive numbers such that $\delta \sum_{k=1}^m t_k^2 \leq 1$. By Proposition 2.12, we need to show that

$$\left\| \sum_{k=1}^m t_k \chi_{T_k} \right\|_{L^2(\mathbb{R}^2)} \lesssim \sqrt{\log(1/\delta)}.$$

Using the preceding lemma we obtain:

$$\begin{aligned} \left\| \sum_{k=1}^m t_k \chi_{T_k} \right\|_{L^2(\mathbb{R}^2)}^2 &= \sum_{k,l=1}^m t_k t_l |T_{e_k}^\delta(a_k) \cap T_{e_l}^\delta(a_l)| \\ &\lesssim \alpha(n-1) \delta \sum_{k=1}^m t_k^2 + \sum_{\substack{k,l=1 \\ k \neq l}}^m t_k t_l \frac{\delta^2}{|e_k - e_l|} \\ &\lesssim \sum_{\substack{k,l=1 \\ k \neq l}}^m \sqrt{\delta} t_k \left(\frac{\delta}{|e_k - e_l|} \right)^{\frac{1}{2}} \sqrt{\delta} t_l \left(\frac{\delta}{|e_k - e_l|} \right)^{\frac{1}{2}}. \end{aligned}$$

By Cauchy-Schwarz,

$$\begin{aligned}
\left\| \sum_{k=1}^m t_k \chi_{T_k} \right\|_{L^2(\mathbb{R}^2)}^2 &\leq \left(\sum_{\substack{k,l=1 \\ k \neq l}}^m \delta t_k^2 \frac{\delta}{|e_k - e_l|} \right)^{\frac{1}{2}} \left(\sum_{\substack{k,l=1 \\ k \neq l}}^m \delta t_l^2 \frac{\delta}{|e_k - e_l|} \right)^{\frac{1}{2}} \\
&= \sum_{\substack{k,l=1 \\ k \neq l}}^m \delta t_k^2 \frac{\delta}{|e_k - e_l|} \\
&\lesssim \log(1/\delta) \sum_{k=1}^m \delta t_k^2 \\
&= \log(1/\delta),
\end{aligned}$$

where the second to last inequality follows from

$$\sum_{\substack{k,l=1 \\ k \neq l}}^m \frac{\delta}{|e_k - e_l|} \lesssim \sum_{1 \leq l \leq \frac{1}{\delta}} \frac{\delta}{l} = \sum_{1 \leq l \leq \frac{1}{\delta}} \frac{1}{l} \approx \frac{1}{\delta}.$$

□

Remark 2.15. If $0 < \delta < 1$, for every $\varepsilon > 0$ there is a constant C_ε such that $\log(1/\delta) \leq C_\varepsilon \delta^{-\varepsilon}$, so the theorem above implies the Keakeya maximal conjecture for $n = 2$.

We are ready to obtain a discrete characterization of the Keakeya maximal inequalities.

Proposition 2.16. *Let $1 < p < \infty$, $q = \frac{p}{p-1} = p'$, $1 \leq M < \infty$ and $0 < \delta < 1$. Then*

$$(9) \quad \|\mathcal{K}_\delta f\|_{L^q(\mathbb{S}^{n-1})} \lesssim_{n,p,\varepsilon} M \delta^{-\varepsilon} \|f\|_{L^p(\mathbb{R}^n)}$$

for all $f \in L^p(\mathbb{R}^n)$ and $\varepsilon > 0$ if and only if

$$(10) \quad \left\| \sum_{k=1}^m \chi_{T_k} \right\|_{L^q(\mathbb{R}^n)} \lesssim_{n,q,\varepsilon} M \delta^{-\varepsilon} (m \delta^{n-1})^{\frac{1}{q}}$$

for all $\varepsilon > 0$ and all δ -separated δ -tubes T_1, \dots, T_m .

Proof. Assume first (10), let T_1, \dots, T_m be δ -separated δ -tubes and t_1, \dots, t_m be positive numbers such that $\delta^{n-1} \sum_{k=1}^m t_k^q \leq 1$. By Proposition 2.12 it suffices to prove that

$$\left\| \sum_{k=1}^m t_k \chi_{T_k} \right\|_{L^q(\mathbb{R}^n)} \lesssim_{n,q,\varepsilon} M \delta^{-\varepsilon}.$$

By Remark 2.10, $m \delta^{n-1} \lesssim 1$. By multiplying (10) by δ^{n-1} and picking ε_0 small such that $\delta^{n-1-\varepsilon_0} < 1$, we get $\left\| \sum_{k=1}^m \delta^{n-1} \chi_{T_k} \right\|_{L^q(\mathbb{R}^n)} \leq C$, where

C is a universal constant. Also, $t_k \leq \delta^{\frac{(1-n)}{q}}$ for all $1 \leq k \leq m$, so it is enough to prove the inequality above for the sum over those k for which $\delta^{n-1} \leq t_k \leq \delta^{\frac{(1-n)}{q}}$. We can break this sum into $\approx \log(1/\delta)$ sums over $I_j = \{k : 2^{j-1} \leq t_k < 2^j\}$ and let m_j be the cardinality of I_j . Observe that for each I_j we have $2^{jq} \leq (2t_k)^q$ for all $t_k \in I_j$, so $m_j 2^{jq} \leq \sum_{I_j} (2t_k)^q \leq \sum_{k=1}^m (2t_k)^q \leq 2^q \delta^{1-n}$. Applying this and (10) with $\varepsilon/2$ we obtain:

$$\begin{aligned}
\left\| \sum_{2^{j-1} \leq t_k < 2^j} t_k \chi_{T_k} \right\|_{L^q(\mathbb{R}^n)} &\leq \sum_j \left\| \sum_{k \in I_j} 2^j \chi_{T_k} \right\|_{L^q(\mathbb{R}^n)} \\
&= \sum_j 2^j \left\| \sum_{k \in I_j} \chi_{T_k} \right\|_{L^q(\mathbb{R}^n)} \\
&\lesssim_{n,q,\varepsilon} \sum_j 2^j M \delta^{-\varepsilon/2} (m_j \delta^{n-1})^{\frac{1}{q}} \\
&\lesssim_{n,q,\varepsilon} M \delta^{-\varepsilon/2} \sum_j 1 \\
&\lesssim_{n,q,\varepsilon} M \log(1/\delta) \delta^{-\varepsilon/2} \\
&\lesssim_{n,q,\varepsilon} M \delta^{-\varepsilon},
\end{aligned}$$

where we used Remark 2.15 for $\varepsilon/2$ in the last inequality.

Conversely, assume 9 holds and let T_1, \dots, T_m be δ -separated δ -tubes with directions e_1, \dots, e_m . Let $g \in L^p(\mathbb{S}^{n-1})$ with $\|g\|_{L^p(\mathbb{S}^{n-1})} \leq 1$. Then,

$$\begin{aligned}
\int_{\mathbb{S}^{n-1}} \sum_{k=1}^m \chi_{T_k} g &= \sum_{k=1}^m \int_{T_k} g \lesssim \sum_{k=1}^m \mathcal{K}_\delta g(e_k) \delta^{n-1} \\
&\lesssim \sum_{k=1}^m \int_{\mathbb{S}^{n-1} \cap B(e_k, \delta)} \mathcal{K}_\delta g(e_k) d\sigma^{n-1}(e) \\
&\lesssim \sum_{k=1}^m \int_{\mathbb{S}^{n-1} \cap B(e_k, \delta)} \mathcal{K}_\delta g(e) d\sigma^{n-1}(e) \\
&= \int_{\bigcup_k (\mathbb{S}^{n-1} \cap B(e_k, \delta))} \mathcal{K}_\delta g(e) d\sigma^{n-1}(e) \\
&\leq \|\mathcal{K}_\delta g\|_{L^p(\mathbb{S}^{n-1})} \left| \bigcup_k (\mathbb{S}^{n-1} \cap B(e_k, \delta)) \right|^{\frac{1}{q}} \\
&\lesssim_{n,p,\varepsilon} M \delta^{-\varepsilon} (m \delta^{n-1})^{\frac{1}{q}}.
\end{aligned}$$

Observe that we used Remark 2.11 above. The proposition follows by taking the supremum over all $g \in L^p(\mathbb{S}^{n-1})$ with $\|g\|_{L^p(\mathbb{S}^{n-1})} = 1$. \square

The next proposition improves the last one by considering only maximal sets of δ -separated δ -tubes. Observe that in this case we have $m\delta^{n-1} \approx 1$.

Proposition 2.17. *Let $1 < q < \infty$, $1 \leq M < \infty$ and $0 < \delta < 1$. Then*

$$(11) \quad \left\| \sum_{k=1}^m \chi_{T_k} \right\|_{L^q(\mathbb{R}^n)} \lesssim_{n,q,\varepsilon} M \delta^{-\varepsilon} (m \delta^{n-1})^{\frac{1}{q}}$$

for all $\varepsilon > 0$ and for all δ -separated δ -tubes T_1, \dots, T_m provided

$$(12) \quad \left\| \sum_{k=1}^m \chi_{T_k} \right\|_{L^q(\mathbb{R}^n)} \lesssim_{n,q,\varepsilon} M \delta^{-\varepsilon}$$

for all $\varepsilon > 0$ and for all δ -separated δ -tubes T_1, \dots, T_m .

Proof. Let m_0 be the maximal cardinality of δ -separated δ -tubes in \mathbb{S}^{n-1} , so $m_0 \approx \delta^{1-n}$ by Remark 2.10. For every $1 \leq m \leq m_0$ let $c(m)$ denote the smallest constant such that

$$\left\| \sum_{k=1}^m \chi_{T_k} \right\|_{L^q(\mathbb{R}^n)} \leq c(m)$$

for all δ -separated δ -tubes T_1, \dots, T_m . Set $c(t) = 0$ for $t < 1$, $c(t) = c(m)$ for $m \leq t < m+1$ if $1 \leq m$ and $c(m) = c(m_0)$ for $m > m_0$. Condition 12 gives

$$(13) \quad c(m) \lesssim M \delta^{-\varepsilon}$$

and we need to prove

$$(14) \quad c(m) \lesssim M \delta^{-\varepsilon} (m \delta^{n-1})^{\frac{1}{q}}.$$

For $m \leq m_0$ fixed, there is a δ -separated subset $S \subset \mathbb{S}^{n-1}$ of cardinality m such that the corresponding tubes T_e , $e \in S$, satisfy

$$\left\| \sum_{e \in S} \chi_{T_e} \right\|_{L^q(\mathbb{R}^n)} = c(m).$$

Indeed, consider the function $h : (\mathbb{S}^{n-1})^m \rightarrow \mathbb{R}$ given by $h(e_1, \dots, e_m) = \left\| \sum_{k=1}^m \chi_{T_{e_k}^\delta} \right\|_{L^q(\mathbb{R}^n)}$. h is continuous and defined on a compact, so it attains a maximum at some m -tuple (e_1, \dots, e_m) of unit vectors. The identity above clearly holds for $S = \{e_1, \dots, e_m\}$.

Consider now rotations $g \in O(n)$ of S such that S and $g(S)$ are disjoint, which happens for almost all $g \in O(n)$. Denote by $T_{g(e)}$ the rotated tube $g(T_e)$. Then we also have:

$$\left\| \sum_{e \in S} \chi_{T_{g(e)}} \right\|_{L^q(\mathbb{R}^n)} = c(m).$$

From the inequality $\|f + g\|_q^q \geq \|f\|_q^q + \|g\|_q^q$ for non-negative functions we get:

(15)

$$\left\| \sum_{e \in S \cup g(S)} \chi_{T_e} \right\|_{L^q(\mathbb{R}^n)} \geq \left(\left\| \sum_{e \in S} \chi_{T_e} \right\|_{L^q(\mathbb{R}^n)}^q + \left\| \sum_{e \in g(S)} \chi_{T_e} \right\|_{L^q(\mathbb{R}^n)}^q \right)^{\frac{1}{q}} = 2^{\frac{1}{q}} c(m).$$

Define

$$a(S, g) := \#\{(e, e') \in S \times g(S) : |e - e'| \leq \delta\}.$$

We can also write this as

$$a(S, g) = \sum_{e \in S} \sum_{e' \in S} \chi_{B(0, \delta)}(e - g(e')).$$

By letting g range over $O(n)$ and using the normalized Haar measure $dO(n)$ we get

$$(16) \quad \int_{O(n)} a(S, g) dO(n) = \sum_{e \in S} \sum_{e' \in S} f(e, e'),$$

where

$$f(e, e') = \int_{O(n)} \chi_{B(0, \delta)}(e - g(e')) dO(n).$$

Given $\omega \in \mathbb{S}^{n-1}$, there is some $g' \in O(n)$ such that $\omega = g'(e')$. Denote $F(g) = \chi_{B(0, \delta)}(e - g(e'))$. By right-invariance of the Haar measure on $O(n)$,

$$\int_{O(n)} F(gg') dO(n) = \int_{O(n)} F(g) dO(n) \Rightarrow f(e, \omega) = f(e, e'),$$

so f is constant in the second entry. By integrating both sides of (16) on \mathbb{S}^{n-1} and using Fubini we obtain

$$(17) \quad f(e, e') = \frac{1}{|\mathbb{S}^{n-1}|} \int_{O(n)} \int_{\mathbb{S}^{n-1}} \chi_{B(0, \delta)}(e - g(\omega)) d\sigma(\omega) dO(n).$$

The inner integral can be estimated as follows:

$$\begin{aligned} \int_{\mathbb{S}^{n-1}} \chi_{B(0, \delta)}(e - g(\omega)) d\sigma(\omega) &= \int_{\mathbb{S}^{n-1}} \chi_{B(0, \delta)}(e - \omega) d\sigma(\omega) \\ &= |\{\omega \in \mathbb{S}^{n-1} : |e - \omega| \leq \delta\}| \\ &\lesssim \delta^{n-1}. \end{aligned}$$

Looking back to (17) we conclude that $f(e, e') \lesssim \delta^{n-1}$ uniformly, so (16) becomes

$$\int_{O(n)} a(S, g) dO(n) \lesssim m^2 \delta^{n-1}.$$

That means that we can find $g \in O(n)$ such that $a(S, g) \leq bm^2 \delta^{n-1}$ for some universal constant C .

Write $S \cup g(S) = S_1 \cup S_2$, where

$$S_2 = \{e \in S : \exists e' \in g(S) \text{ such that } |e - e'| \leq \delta\},$$

$$S_1 = (S \setminus S_2) \cup g(S).$$

Observe that c is non-decreasing on natural numbers. Indeed, for any $m+1$ collection of δ -separated δ -tubes we have:

$$\left\| \sum_{k=1}^m \chi_{T_k} \right\|_{L^q(\mathbb{R}^n)} \leq \left\| \sum_{k=1}^{m+1} \chi_{T_k} \right\|_{L^q(\mathbb{R}^n)} \leq c(m+1).$$

Since any collection of m δ -separated δ -tubes can be put in a collection of $m+1$ of these (for $m < m_0$), $c(m) \leq c(m+1)$ by definition. By the extension we defined in the beginning of the proof, c is non-decreasing. For S_1 we have the trivial bound $\#S_1 \leq 2m$ and for S_2 we have $\#S_2 \leq bm^2\delta^{n-1}$. By the triangle inequality,

$$(18) \quad \left\| \sum_{e \in S \cup g(S)} \chi_{T_e} \right\|_{L^q(\mathbb{R}^n)} \leq c(2m) + c(bm^2\delta^{n-1}).$$

Combining (18) with (15) we get

$$2^{\frac{1}{q}}c(m) \leq c(2m) + c(bm^2\delta^{n-1}).$$

Up to multiplying by a constant less than 2, we can consider $b\delta^{n-1} = 2^{-k}$ for some $k \in \mathbb{N}$. It is enough to prove (14) for $m = 2^{N-k}$, $k = 1, \dots, N$. In fact, we know that for 2^l with $l \geq N$ this inequality reduces to (13), and for the other integers we can just use the monotonicity of c to compare $c(m)$ to $c(2^l)$ where $2^{l-1} < m < 2^l$.

We have $m = 2^{-k}b^{-1}\delta^{1-n}$. Set

$$c_k = 2^{\frac{k}{q}}c(2^{-k}b^{-1}\delta^{1-n}), \quad k = 1, \dots, N.$$

Then the last inequality becomes

$$(19) \quad c_k \leq c_{k-1} + 2^{-\frac{(k+1)}{q}}c_{2k}.$$

Inequality (14) now becomes $c_k \lesssim_{n,q,\varepsilon} M\delta^{-\varepsilon}$. If $k \geq B \log(1/\delta^{n-1})$ for B big enough, $m \leq 1/2$ and $c(m) = 0$. Thus it suffices to restrict ourselves to $k \lesssim \log(1/\delta)$. Define a new sequence d_k by

$$(20) \quad d_k = (1 + C_0 2^{-\frac{k}{q}})c_k,$$

for some constant C_0 to be chosen later. We claim that if $A > 1$ then

$$(21) \quad d_k < d_{k-1} + 2^{-\frac{k}{q}}(A(d_{2k} - d_k) + (d_{k-1} - d_k))$$

for all $k \geq k_0$, where k_0 is some fixed constant. To verify this, multiply (19) by $(1 + C_0 2^{-\frac{k}{q}})$ to find that

$$(22) \quad \begin{aligned} d_k &\leq c_{k-1}(1 + C_0 2^{-\frac{k}{q}}) + 2^{-\frac{k}{q}} c_{2k}(1 + C_0 2^{-\frac{k}{q}}) \\ &= d_{k-1} \left(\frac{1 + C_0 2^{-\frac{k}{q}}}{1 + C_0 2^{-\frac{k-1}{q}}} \right) + C 2^{-\frac{k}{q}} d_{2k} \left(\frac{1 + C_0 2^{-\frac{k}{q}}}{1 + C_0 2^{-\frac{k-1}{q}}} \right). \end{aligned}$$

Subtracting (20) from (22) and rearranging we see that we have to show that

$$(A+1)2^{-\frac{k}{q}} d_k < d_{k-1} \left(1 + 2^{-\frac{k}{q}} - \frac{1 + C_0 2^{-\frac{k}{q}}}{1 + C_0 2^{-\frac{k-1}{q}}} \right) + d_{2k} \left(A - \frac{1 + C_0 2^{-\frac{k}{q}}}{1 + C_0 2^{-\frac{k-1}{q}}} \right).$$

By substituting in for c_k we conclude that our goal is to show that

$$c_k < c_{k-1} \left[\frac{1 + C_0(2^{\frac{1}{q}} - 1) + C_0 2^{-\frac{(k-1)}{q}}}{(A+1)(1 + C_0 2^{-\frac{k}{q}})} \right] + 2^{-\frac{k}{q}} \left[\frac{(A-1)2^{\frac{k}{q}} - C_0 + A C_0 2^{-\frac{k}{q}}}{(A+1)(1 + C_0 2^{-\frac{k}{q}})} \right].$$

Since we know that (19) holds, it is enough to show that each term in the brackets above is greater than 1. For the first term, notice that for all large k we have

$$\frac{1 + C_0(2^{\frac{1}{q}} - 1) + C_0 2^{-\frac{(k-1)}{q}}}{(A+1)(1 + C_0 2^{-\frac{k}{q}})} > \frac{1 + C_0(2^{\frac{1}{q}} - 1)}{2(A+1)},$$

and this can be made greater than 1 by choosing C_0 big enough. For the second term we have

$$\frac{(A-1)2^{\frac{k}{q}} - C_0 + A C_0 2^{-\frac{k}{q}}}{(A+1)(1 + C_0 2^{-\frac{k}{q}})} > \frac{(A-1)2^{\frac{k}{q}} - C_0}{2(A+1)}.$$

Since $A > 1$, this can also be made greater than 1 by choosing k big enough, so choose k_0 such that both bounds hold for all $k \geq k_0$ and the claim is proved. Let $d_{k_{\max}}$ be the maximal of d_k in the range $[k_0, B \log(1/\delta)]$. Applying (21) we have:

$$d_{k_{\max}} < d_{k_{\max}-1} + 2^{-\frac{k_{\max}}{q}} (A(d_{2k_{\max}} - d_{k_{\max}}) + (d_{k_{\max}-1} - d_{k_{\max}})).$$

By the maximality of $d_{k_{\max}}$, the second summand on the right-hand side is negative, so $d_{k_{\max}} < d_{k_{\max}-1}$, which forces $k_{\max} = k_0$. Therefore $c_k \lesssim c_{k_0}$ for all $k \geq k_0$. Notice that we trivially have $c_k \lesssim c_{k_0}$ for all $k < k_0$. Since $c_{k_0} = 2^{\frac{k_0}{q}} c(2^{-k} b^{-1} \delta^{1-n}) \lesssim c(\delta^{1-n}) \approx c_0 \lesssim 1$ by hypothesis, we have $c_k \lesssim 1$ for all $0 \leq k \lesssim \log(1/\delta)$, and this concludes the proof of this proposition. \square

Combining these last two propositions we have:

Corollary 2.18. *Let $1 < p < \infty$, $q = \frac{p}{p-1}$, $0 < \beta < \infty$ and $0 < \delta < 1$. Then*

$$\|\mathcal{K}_\delta f\|_{L^p(\mathbb{S}^{n-1})} \lesssim_{n,p,\varepsilon} \delta^{-\beta-\varepsilon} \|f\|_{L^p(\mathbb{S}^{n-1})}$$

for all $f \in L^p(\mathbb{S}^{n-1})$, $\varepsilon > 0$, if and only if

$$\left\| \sum_{k=1}^m \chi_{T_k} \right\|_{L^q(\mathbb{R}^n)} \lesssim_{n,p,\varepsilon} \delta^{-\beta-\varepsilon}$$

for all $\varepsilon > 0$ and for all δ -separated δ -tubes T_1, \dots, T_m . In particular, the Kakeya maximal conjecture 2.7 holds if and only if

$$\left\| \sum_{k=1}^m \chi_{T_k} \right\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \lesssim_{n,p,\varepsilon} \delta^{-\varepsilon}$$

for all $\varepsilon > 0$ and for all δ -separated δ -tubes T_1, \dots, T_m .

3. KAKEYA MAXIMAL IMPLIES KAKEYA

Theorem 3.1. *Suppose that $0 < \delta < 1$, $\beta > 0$ and $n - \beta p > 0$. If*

$$\|\mathcal{K}_\delta f\|_{L^p(\mathbb{S}^{n-1})} \leq C(n, p, \beta) \delta^{-\beta} \|f\|_{L^p(\mathbb{R}^n)}$$

for all $0 < \delta < 1$ and $f \in L^p(\mathbb{R}^n)$, then the Hausdorff dimension of every Besicovitch set in \mathbb{R}^n is at least $n - \beta p$. In particular, if (4) holds for some p , then the Hausdorff dimension of every Besicovitch set in \mathbb{R}^n is n . Thus Conjecture 2.7 implies the Kakeya conjecture 1.4.

Proof. Let $B \subset \mathbb{R}^n$ be a Besicovitch set. Let $0 < \alpha < n - \beta p$ and $B_j = B(x_j, r_j)$ be a collection of balls such that $B \subset \cup_j B_j$ and $r_j < 1$. By the equivalent characterization of Hausdorff dimension we stated in section 1, it suffices to show that $\sum_j r_j^\alpha \gtrsim 1$. For $e \in \mathbb{S}^{n-1}$ let I_e be a unit segment parallel to e . For $k = 1, 2, \dots$, set

$$J_k := \{j : 2^{-k} \leq r_j < 2^{1-k}\},$$

and

$$S_k := \left\{ e \in \mathbb{S}^{n-1} : \mathcal{H}^1 \left(I_e \cap \bigcup_{j \in J_k} B_j \right) \geq \frac{1}{2k^2} \right\}.$$

We claim that

$$(23) \quad \mathbb{S}^{n-1} = \bigcup_k S_k.$$

Indeed, if there was $e \in \mathbb{S}^{n-1} \setminus \cup_k S_k$, then $\mathcal{H}^1(I_e \cap (\cup_{j \in J_k} B_j)) < 1/2k^2$ for all k , so

$$\sum_k \mathcal{H}^1 \left(I_e \cap \bigcup_{j \in J_k} B_j \right) < \sum_k \frac{1}{2k^2} < 1.$$

On the other hand,

$$1 \leq \mathcal{H}^1(I_e) \leq \sum_k \mathcal{H}^1 \left(I_e \cap \bigcup_{j \in J_k} B_j \right),$$

contradiction. So (23) holds. Let

$$f_k = \chi_{F_k} \text{ with } F_k = \bigcup_{j \in J_k} B(x_j, 2r_j).$$

For $e \in S_k$, let a_e be the midpoint of I_e . We claim that

$$(24) \quad |T_e^{2^{-k}}(a_e) \cap F_k| \gtrsim \frac{1}{k^2} |T_e^{2^{-k}}(a_e)|.$$

To prove this, we first show that $\forall y \in \bigcup_{j \in J_k} B(x_j, r_j) := M_k$ we have $F_k \supseteq B(y, 2^{-k})$. To verify this pick such y , so $y \in B(x_j, r_j)$ for some $j \in J_k$. Let $z \in B(y, 2^{-k})$. Then

$$|x_j - z| \leq |x_j - y| + |y - z| \leq r_j + 2^{-k} \leq 2r_j,$$

so $z \in B(x_j, 2r_j) \subset F_k$. This way,

$$\begin{aligned} T_e^{2^{-k}}(a_e) \cap F_k &\supseteq \bigcup_{y \in M_k} \left(T_e^{2^{-k}}(a_e) \cap B(y, 2^{-k}) \right) \\ &= T_e^{2^{-k}}(a_e) \cap \bigcup_{y \in M_k} B(y, 2^{-k}) \\ &\supseteq T_e^{2^{-k}}(a_e) \cap \bigcup_{y \in I_e \cap M_k} B(y, 2^{-k}). \end{aligned}$$

Observe that $T_e^{2^{-k}}(a_e) \cap \bigcup_{y \in I_e \cap M_k} B(y, 2^{-k})$ is exactly the portion of $T_e^{2^{-k}}(a_e)$ that shadows $I_e \cap M_k$, so it has measure at least $\gtrsim |I_e \cap M_k| 2^{-k(n-1)} \gtrsim |T_e^{2^{-k}}(a_e)|/k^2$. This shows (24), which implies $\mathcal{K}_{2^{-k}} f_k(e) \gtrsim 1/k^2$ for all $e \in S_k$. This and the Kakeya maximal conjecture imply:

$$\begin{aligned} \sigma^{n-1}(S_k) &\lesssim k^{2p} \int_{S_k} |\mathcal{K}_{2^{-k}} f_k(e)|^p d\sigma(e) \leq k^{2p} \int_{\mathbb{S}^{n-1}} |\mathcal{K}_{2^{-k}} f_k(e)|^p d\sigma(e) \\ &\leq k^{2p} C_{n,p,\beta}^p 2^{k\beta p} \int_{\mathbb{R}^n} |f_k(x)|^p dx \\ &= k^{2p} C_{n,p,\beta}^p 2^{k\beta p} |F_k|. \end{aligned}$$

On the other hand, $|F_k| \leq \#J_k \alpha(n) (2r_j)^n < \#J_k \alpha(n) 2^{(2-k)n}$, so

$$\sigma^{n-1}(S_k) \lesssim k^{2p} 2^{k\beta p} 2^{-kn} 2^{2n} \#J_k = k^{2p} 2^{k[\alpha(n) - (n-\beta p)]} 2^{-k\alpha} \#J_k \lesssim 2^{-k\alpha} \#J_k,$$

since $\lim_{k \rightarrow \infty} k^{2p} 2^{k[\alpha(n) - (n-\beta p)]} = 0$. Finally,

$$\sum_j r_j^\alpha \gtrsim \sum_k \#J_k 2^{-k\alpha} \gtrsim \sum_k \sigma^{n-1}(S_k) \gtrsim 1.$$

This concludes the proof. \square

Now we give a different proof of Theorem 2.14. The proof is general enough to provide sharp L^2 estimates in \mathbb{R}^n . For this we will need the following lemma:

Lemma 3.2 (Bourgain). *Let $C \geq 1$ be some constant and suppose $0 < \delta < 1$. For any fixed $\xi \in \mathbb{R}^n$ one has the bound*

$$(25) \quad |\{e \in \mathbb{S}^{n-1} : \xi \in C \cdot T_e^{1/\delta}(0)\}| \lesssim \frac{1}{1+|\xi|}.$$

Proof. First suppose that $|\xi| < 10C$ so that $\frac{1}{1+10C} < \frac{1}{1+|\xi|}$. It follows that

$$|\{e \in \mathbb{S}^{n-1} : \xi \in C \cdot T_e^{1/\delta}(0)\}| \leq |\mathbb{S}^{n-1}| \lesssim \frac{1}{1+10C} < \frac{1}{1+|\xi|}.$$

If $|\xi| > \frac{C}{\delta}$, then $\xi \notin C \cdot T_e^{1/\delta}(0)$ for all $e \in \mathbb{S}^{n-1}$ by definition and the estimate above is trivial.

It remains to prove it now for $10C \leq |\xi| \leq \frac{C}{\delta}$. By definition, we have:

$$C \cdot T_e^{1/\delta}(0) := \{\eta \in \mathbb{R}^n : |\eta \cdot e| \leq C/2, |\eta \cdot v| \leq C/\delta \ \forall \text{ unit } v \perp e\}.$$

We may rotate the coordinate axes so that $\xi = (\xi_1, 0, \dots, 0)$. Notice that

$$\begin{aligned} |\{e \in \mathbb{S}^{n-1} : \xi \in C \cdot T_e^{1/\delta}(0)\}| &= |\{e \in \mathbb{S}^{n-1} : |\xi \cdot e| \leq C/2, |\xi \cdot v| \leq C/\delta \ \forall \text{ unit } v \perp e\}| \\ &= \left| \left\{ e \in \mathbb{S}^{n-1} : \left| \frac{\xi}{|\xi|} \cdot e \right| \leq \frac{C}{2|\xi|}, \left| \frac{\xi}{|\xi|} \cdot v \right| \leq \frac{C}{\delta|\xi|} \ \forall \text{ unit } v \perp e \right\} \right| \\ &= \left| \left\{ e \in \mathbb{S}^{n-1} : \left| \frac{\xi}{|\xi|} \cdot e \right| \leq \frac{C}{2|\xi|} \right\} \right|, \end{aligned}$$

since

$$\left| \frac{\xi}{|\xi|} \cdot e \right| \leq \frac{C}{2|\xi|} \Rightarrow \left| \frac{\xi}{|\xi|} \cdot v \right| \leq \left| \frac{\xi}{|\xi|} \right| |v| = 1 \leq \frac{C}{\delta|\xi|},$$

for all $v \perp e$ since $|\xi| \leq \frac{C}{\delta}$.

Now $\left| \frac{\xi}{|\xi|} \cdot e \right| = |e_1|$ where $e = (e_1, e_2, \dots, e_n)$, so

$$|\{e \in \mathbb{S}^{n-1} : \xi \in C \cdot T_e^{1/\delta}(0)\}| = \left| \left\{ e \in \mathbb{S}^{n-1} : |e_1| \leq \frac{C}{2|\xi|} \right\} \right|.$$

This defines a subset of \mathbb{S}^{n-1} that looks like a ring of thickness $\approx \frac{1}{|\xi|}$ in the e_1 direction. Thus the size of this ring is bounded by $\frac{1}{|\xi|} \lesssim \frac{1}{1+|\xi|}$ since $|\xi| \gtrsim 1$ and the lemma follows. \square

Theorem 3.3. *For all $0 < \delta < 1$ and $f \in L^2(\mathbb{R}^2)$,*

$$\|\mathcal{K}_\delta f\|_{L^2(\mathbb{S}^1)} \leq C \sqrt{\log(1/\delta)} \|f\|_{L^2(\mathbb{R}^2)},$$

with some absolute constant C . In \mathbb{R}^n , $n \geq 3$, we have for all $0 < \delta < 1$ and $f \in L^2(\mathbb{R}^n)$:

$$\|\mathcal{K}_\delta f\|_{L^2(\mathbb{S}^{n-1})} \leq C(n) \delta^{(2-n)/2} \|f\|_{L^2(\mathbb{R}^n)},$$

where the exponent $(2-n)/2$ is the best possible.

Proof. We may assume f to be non-negative since $\mathcal{K}_\delta f = \mathcal{K}_\delta |f|$. We can also assume $f \in C_c^\infty(\mathbb{R}^n)$ since this is dense in L^2 . Let $\varphi_e^\delta(x) := \frac{1}{\alpha(n-1)\delta^{n-1}} \chi_{T_e^\delta(0)}(x)$. Then, by a change of variables and by the symmetry of $T_e^\delta(0)$ we have

$$\begin{aligned} \mathcal{K}_\delta f(e) &= \sup_{a \in \mathbb{R}^n} \frac{1}{\alpha(n-1)\delta^{n-1}} \int_{T_e^\delta(a)} f(y) dy \\ &= \sup_{a \in \mathbb{R}^n} \frac{1}{\alpha(n-1)\delta^{n-1}} \int_{T_e^\delta(0)} f(y-a) dy \\ &= \sup_{a \in \mathbb{R}^n} \frac{1}{\alpha(n-1)\delta^{n-1}} \int_{\mathbb{R}^n} \chi_{T_e^\delta(0)}(y) f(a-y) dy \\ &= \sup_{a \in \mathbb{R}^n} (\varphi_e^\delta * f)(a). \end{aligned}$$

Let $\phi \in \mathcal{S}(\mathbb{R})$ such that $\phi \geq 0$, $\text{supp}(\hat{\phi}) \subset [-\frac{C}{2}, \frac{C}{2}]$ and $\phi(x) \geq 1$ when $|x| \leq \frac{1}{2}$. Define $\psi : \mathbb{R}^n \rightarrow \mathbb{R}$ by

$$\psi(x) = \phi(x_1) \frac{1}{\delta^{n-1}} \prod_{j=2}^n \phi\left(\frac{x_j}{\delta}\right), \quad x = (x_1, \dots, x_n).$$

Let $e_1 = (1, 0, \dots, 0) \in \mathbb{R}^n$. Observe that $\varphi_{e_1}^\delta \leq \psi$. Indeed, if $x \notin T_{e_1}^\delta(0)$ then $\varphi_{e_1}^\delta(x) = 0 \leq \psi(x)$. If $x \in T_{e_1}^\delta(0)$ then $\phi(x_1) \geq 1$ and $\phi(x_j/\delta) \geq 1$ and we are also done. This way,

$$\mathcal{K}_\delta f(e_1) \leq \sup_{a \in \mathbb{R}^n} (\psi * f)(a)$$

Let ρ_e be a rotation that takes e to e_1 . Define $\psi_e = \psi \circ \rho_e$. As before, $\varphi_e^\delta \leq \psi_e$, so

$$\varphi_e^\delta * f \leq \psi_e * f \Rightarrow \mathcal{K}_\delta f(e) = \sup_{a \in \mathbb{R}^n} (\varphi_e^\delta * f) \leq \sup_{a \in \mathbb{R}^n} (\psi_e * f) = \|\psi_e * f\|_\infty.$$

By Fourier inversion, since $\psi_e * f$ and $\widehat{\psi_e * f}$ are in $L^1(\mathbb{R}^n)$,

$$\|\psi_e * f\|_\infty \leq \|\widehat{\psi_e * f}\|_{L^1(\mathbb{R}^n)} = \|\widehat{\psi_e} \widehat{f}\|_{L^1(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |\widehat{\psi_e}(\xi)| \cdot |\widehat{f}(\xi)| d\xi.$$

Thus by Cauchy-Schwarz we have

$$\begin{aligned} \mathcal{K}_\delta f(e) &\leq \int_{\mathbb{R}^n} |\widehat{f}(\xi)| \sqrt{|\widehat{\psi_e}(\xi)| (1 + |\xi|)} \sqrt{\frac{|\widehat{\psi_e}(\xi)|}{1 + |\xi|}} d\xi \\ (26) \quad &\leq \left(\int_{\mathbb{R}^n} |\widehat{f}(\xi)|^2 |\widehat{\psi_e}(\xi)| (1 + |\xi|) d\xi \right)^{\frac{1}{2}} \underbrace{\left(\int_{\mathbb{R}^n} \frac{|\widehat{\psi_e}(\xi)|}{1 + |\xi|} d\xi \right)^{\frac{1}{2}}}_{(*)} \end{aligned}$$

Let us estimate $(*)$ now. We know that $\widehat{\psi}_e = \widehat{\psi} \circ \rho_e$. By a direct computation we get:

$$\widehat{\psi}(\xi) = \widehat{\phi}(\xi_1) \prod_{j=2}^n \widehat{\phi}(\delta \xi_j).$$

By our assumptions on $\widehat{\phi}$ we see that $\widehat{\psi}$ is supported on a tube of size $C(1 \times 1/\delta \times \dots 1/\delta)$ oriented in the direction of e_1 . It then follows that $\widehat{\psi}_e$ is bounded and supported on a tube of size $C(1 \times 1/\delta \times \dots 1/\delta)$ oriented in the direction of e , i.e. $\widehat{\psi}_e$ is supported on $C \cdot T_e^{1/\delta}(0)$. This way,

$$\int_{\mathbb{R}^n} \frac{|\widehat{\psi}_e(\xi)|}{1 + |\xi|} d\xi \lesssim \int_{C \cdot T_e^{1/\delta}(0)} \frac{1}{1 + |\xi|} d\xi = \int_{C \cdot T_{e_1}^{1/\delta}(0)} \frac{1}{1 + |\xi|} d\xi$$

by performing a rotation on the region of integration. Denote $\xi = (\xi_1, \xi')$. If $n = 2$ we have:

$$\begin{aligned} \int_{|\xi_1| \leq C, |\xi'| \leq \frac{C}{\delta}} \frac{1}{1 + |\xi|} d\xi_1 d\xi' &\lesssim \int_{|\xi'| \leq \frac{C}{\delta}} \frac{1}{1 + |\xi'|} d\xi' \\ (27) \quad &\lesssim \int_{|\eta| \leq \frac{1}{\delta}} \frac{1}{1 + |\eta|} d\eta \\ &\lesssim \log(1/\delta). \end{aligned}$$

Combining (26) with (27) we obtain:

$$\|\mathcal{K}_\delta f\|_{L^2(\mathbb{S}^1)}^2 \lesssim \sqrt{\log(1/\delta)} \int_{\mathbb{R}^n} |\widehat{f}(\xi)|^2 (1 + |\xi|) \left(\int_{\mathbb{S}^1} |\widehat{\psi}_e(\xi)| d\xi \right) d\xi.$$

Applying Lemma 3.2:

$$\int_{\mathbb{S}^1} |\widehat{\psi}_e(\xi)| d\xi \lesssim |\text{supp}(\widehat{\psi}_e(\xi))| \leq |\{e \in \mathbb{S}^{n-1} : \xi \in C \cdot T_e^{1/\delta}(0)\}| \lesssim \frac{1}{1 + |\xi|},$$

so

$$\|\mathcal{K}_\delta f\|_{L^2(\mathbb{S}^1)}^2 \lesssim \log(1/\delta) \int_{\mathbb{R}^n} |\widehat{f}(\xi)|^2 (1 + |\xi|) \frac{1}{(1 + |\xi|)} d\xi = \log(1/\delta) \|f\|_{L^2(\mathbb{R}^2)}^2$$

by Plancherel.

If $n \geq 3$, by polar coordinates we get

$$\begin{aligned} \int_{|\xi_1| \leq C, |\xi'| \leq \frac{C}{\delta}} \frac{1}{1 + |\xi|} d\xi_1 d\xi' &\lesssim \int_{|\xi'| \leq \frac{C}{\delta}} \frac{1}{1 + |\xi'|} d\xi' \\ (28) \quad &\lesssim \int_{-\frac{C}{\delta}}^{\frac{C}{\delta}} \frac{r^{n-2}}{r} dr \\ &\lesssim \delta^{-(n-2)}. \end{aligned}$$

Combining (26) with (28) and using again Lemma 3.2 we obtain

$$\|\mathcal{K}_\delta f\|_{L^2(\mathbb{S}^{n-1})}^2 \lesssim \delta^{-(n-2)} \int_{\mathbb{R}^n} |\widehat{f}(\xi)|^2 (1 + |\xi|) \frac{1}{(1 + |\xi|)} d\xi = \delta^{-(n-2)} \|f\|_{L^2(\mathbb{R}^n)}^2.$$

Taking square roots, we are done. The power $\delta^{(2-n)/2}$ is the best possible by Remark 2.6. \square

Combining Theorems 3.1 and 3.3 we conclude the following:

Corollary 3.4. *All Besicovitch sets in \mathbb{R}^n , $n \geq 2$, have Hausdorff dimension at least 2.*

4. RESTRICTION IMPLIES KAKEYA

We start this section by stating the restriction conjecture for the sphere:

Conjecture 4.1 (Restriction). *For $q > \frac{2n}{n-1}$ we have*

$$\|\widehat{f}\|_{L^q(\mathbb{R}^n)} \lesssim_{n,q} \|f\|_{L^q(\mathbb{S}^{n-1})}$$

for $f \in L^q(\mathbb{S}^{n-1})$.

Theorem 4.2. *Suppose $\frac{2n}{n-1} < q < \infty$ and*

$$(29) \quad \|\widehat{f}\|_{L^q(\mathbb{R}^n)} \lesssim_{n,q} \|f\|_{L^q(\mathbb{S}^{n-1})}$$

for $f \in L^q(\mathbb{S}^{n-1})$. Then with $p = \frac{q}{q-2}$,

$$(30) \quad \|\mathcal{K}_\delta f\|_{L^p(\mathbb{S}^{n-1})} \lesssim_{n,q} \delta^{\frac{4n}{q}-2(n-1)} \|f\|_{L^p(\mathbb{R}^n)}$$

for all $0 < \delta < 1$ and $f \in L^p(\mathbb{R}^n)$. In particular, the restriction conjecture 4.1 implies the Kakeya maximal conjecture 2.7.

Proof. Let us prove the second statement assuming the first one is true. Observe that $2(n-1) - \frac{4n}{q} \rightarrow 0$ as $q \rightarrow \frac{2n}{n-1}$. Hence for any $\varepsilon > 0$ there is $q > \frac{2n}{n-1}$ such that $2(n-1) - \frac{4n}{q} < \varepsilon$. Then $p = \frac{q}{q-2} < n$ and

$$\|\mathcal{K}_\delta f\|_{L^p(\mathbb{S}^{n-1})} \lesssim_{n,q} \delta^{-\varepsilon} \|f\|_{L^p(\mathbb{R}^n)}$$

for all $f \in L^p(\mathbb{R}^n)$. Interpolating this with the trivial inequality (1) we get

$$\|\mathcal{K}_\delta f\|_{L^n(\mathbb{S}^{n-1})} \lesssim_{n,q} \delta^{-\varepsilon} \|f\|_{L^n(\mathbb{R}^n)}$$

for all $0 < \delta < 1$ and $f \in L^p(\mathbb{R}^n)$, as required.

To prove the first part, let $p' = \frac{p}{p-1} = \frac{q}{2}$, $\{e_1, \dots, e_m\} \subset \mathbb{S}^{n-1}$ be a δ -separated subset, $a_1, \dots, a_m \in \mathbb{R}^n$ and $t_1, \dots, t_m > 0$ with

$$\delta^{n-1} \sum_{k=1}^m t_k^{p'} \leq 1,$$

and let $T_k = T_{e_k}^\delta(a_k)$. We shall show that

$$(31) \quad \left\| \sum_{k=1}^m t_k \chi_{T_k} \right\|_{L^{p'}(\mathbb{R}^n)} \lesssim \delta^{\frac{4n}{q}-2(n-1)}.$$

By Proposition 2.12 this implies (30).

Let τ_k be the δ^{-2} dilation of T_k : it is centered at $\delta^{-2}a_k$, it has length δ^{-2} and cross-section radius δ^{-1} . Let

$$S_k := \{e \in \mathbb{S}^{n-1} : 1 - e \cdot e_k \leq C^{-2}\delta^2\}.$$

Then S_k is a spherical cap of radius $\approx C^{-1}\delta$ and centre e_k . Here C is chosen big enough to guarantee that the S_k are disjoint. Define f_k by

$$f_k(x) = e^{2\pi i \delta^{-2} a_k \cdot x} \chi_{S_k}(x).$$

Then $\|f_k\|_\infty = 1$, $\text{supp}(f_k) \subset S_k$ and $\widehat{f_k}(\xi) = \widehat{\chi_{S_k}}(\xi - \delta^{-2}a_k)$. Provided that C is large enough, but still depending only on n , Knapp's example gives

$$|\widehat{f_k}(\xi)| \gtrsim \delta^{n-1}$$

for $\xi \in \tau_k$.

Fix $s_k \geq 0$, $k = 1, \dots, m$. For $\omega \in \{-1, 1\}^m$ let

$$f_\omega = \sum_{k=1}^m \omega_k s_k f_k.$$

We shall consider the ω_k as independent random variables taking values 1 and -1 with equal probability. Since the functions f_k have disjoint supports,

$$(32) \quad \|f_\omega\|_{L^q(\mathbb{S}^{n-1})}^q = \sum_{k=1}^m \|s_k f_k\|_{L^q(\mathbb{S}^{n-1})}^q \approx \sum_{k=1}^m s_k^q \delta^{n-1}.$$

By Fubini's theorem and Khintchine's inequality,

$$(33) \quad \begin{aligned} \mathbb{E} \left(\|\widehat{f_\omega}\|_{L^q(\mathbb{R}^n)}^q \right) &= \int \mathbb{E}(|\widehat{f_\omega}(\xi)|^q) d\xi \approx \int \left(\sum_{k=1}^m s_k^2 |\widehat{f_k}(\xi)|^2 \right)^{\frac{q}{2}} d\xi \\ &\gtrsim \delta^{q(n-1)} \int \left(\sum_{k=1}^m s_k^2 \chi_{\tau_k}(\xi) \right)^{\frac{q}{2}} d\xi, \end{aligned}$$

since $|\widehat{f_k}| \gtrsim \delta^{n-1} \chi_{\tau_k}$.

By assumption, (29) holds and we have

$$(34) \quad \|\widehat{f_\omega}\|_{L^q(\mathbb{R}^n)} \lesssim_{n,q} \|f_\omega\|_{L^q(\mathbb{S}^{n-1})}.$$

Combining (32), (33) and (34) we obtain:

$$\begin{aligned}
 (35) \quad \delta^{q(n-1)} \int \left(\sum_{k=1}^m s_k^2 \chi_{T_k}(\xi) \right)^{\frac{q}{2}} d\xi &\lesssim \mathbb{E} \left(\|\widehat{f_\omega}\|_{L^q(\mathbb{R}^n)}^q \right) \\
 &\lesssim \mathbb{E} \left(\|f_\omega\|_{L^q(\mathbb{S}^{n-1})}^q \right) \\
 &\lesssim \mathbb{E} \left(\sum_{k=1}^m s_k^q \delta^{n-1} \right) \\
 &\lesssim \sum_{k=1}^m s_k^q \delta^{n-1}.
 \end{aligned}$$

Now we choose $s_k = \sqrt{t_k}$ and have

$$\delta^{n-1} \sum_{k=1}^m s_k^q = \delta^{n-1} \sum_{k=1}^m t_k^{p'} \leq 1.$$

Thus going back to (35) and making the change $y = \delta^2 x$, τ_k goes to T_k and

$$\delta^{q(n-1)} \delta^{-2n} \int \left(\sum_{k=1}^m t_k \chi_{T_k} \right)^{p'} \lesssim 1,$$

which is precisely (31), as required. \square

Since the Kakeya maximal conjecture implies the Kakeya conjecture, we have:

Corollary 4.3. *The restriction conjecture 4.1 implies the Kakeya conjecture 1.4.*

Combining Theorems 3.1 and 4.2, we obtain:

Corollary 4.4. *If $\frac{2n}{n-1} < q < \infty$ and*

$$\|\widehat{f}\|_{L^q(\mathbb{R}^n)} \leq C(n, q) \|f\|_{L^q(\mathbb{S}^{n-1})}$$

for $f \in L^q(\mathbb{S}^{n-1})$, then $\dim B \geq \frac{2n-(n-2)q}{q-2}$ for every Besicovitch set B in \mathbb{R}^n .

5. NIKODYM MAXIMAL FUNCTION

Definition 5.1. A *Nikodym set* is a Borel subset $N \subset \mathbb{R}^n$ of measure zero such that for every $x \in \mathbb{R}^n$ there is a L containing x such that $L \cap N$ contains a unit line segment.

More about Nikodym can be found in [4] on page 147. The related maximal function of a locally integrable function f is the *Nikodym maximal function*:

Definition 5.2. For $0 < \delta < 1$ define:

$$\mathcal{N}_\delta f(x) = \sup_{T \ni x} \frac{1}{|T|} \int_T |f(y)| dy, \quad x \in \mathbb{R}^n.$$

where the supremum is taken over all tubes $T = T_e^\delta(a)$ containing x .

Conjecture 5.3 (Nikodym maximal conjecture). *For all $\varepsilon > 0$, $0 < \delta < 1$:*

$$\|\mathcal{N}_\delta f\|_{L^n(\mathbb{R}^n)} \leq C(n, \varepsilon) \delta^{-\varepsilon} \|f\|_{L^n(\mathbb{R}^n)}.$$

Theorem 5.4 (Tao). *The Keakeya maximal conjecture 2.7 and the Nikodym maximal conjecture 5.3 are equivalent.*

Proof. See [2]. □

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