Vectors and Dot Product

Name: ____________

Math 1920 - Sections 221 and 222 - TA: Itamar Oliveira

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Review

(1) A *vector* $\mathbf{v}=\overrightarrow{PQ}$ is determined by a basepoint P and a terminal point Q.

(2) Components of $\mathbf{v}=\overrightarrow{PQ}$, where $P = (a_1, b_1)$ and $Q = (a_2, b_2)$:

$$\mathbf{v} = \langle a, b\rangle$$

with $a = a_2 - a_1$ and $b = b_2 - b_1$.

(3) The length $\|\mathbf{v}\|$ of $\mathbf{v}$ is equal to ________ (1).

(4) *Vector addition:* $\langle v_1, v_2\rangle + \langle w_1, w_2\rangle =$ ________ (2).

(5) *Scalar multiplication:* $\|\lambda\mathbf{v}\| = |\lambda|\|\mathbf{v}\|$ for λ real.

(6) $\mathbf{v}$ and $\mathbf{w}$ are *parallel* if, for some scalar λ, ________ (3).

(7) If $\mathbf{v}$ makes an angle θ with the positive x-axis, then $v_1=$ ________ (4) and $v_2=$ ________ (5).

(8) *Equation of the sphere of radius R and center (a,b,c):* ________ (6).

(9) *Equation of the cylinder of radius R and vertical axis through (a,b,0):* ________ (7).

(10) *Equations for the line passing through $P_0 = (x_0, y_0, z_0)$ with direction vector $\mathbf{v} = \langle a, b, c\rangle$:*

(a) *Vector parametrization:* $\mathbf{r}(t) = \overrightarrow{OP_0} + t\mathbf{v} =$ ________ (8).

(b) *Parametric equation:* $x =$ ________ (9), $y =$ ________ (10), $z =$ ________ (11).

(11) The *dot product* of $\mathbf{v} = \langle a_1, b_1, c_1\rangle$ and $\mathbf{w} = \langle a_2, b_2, c_2\rangle$ is

$$\mathbf{v}\cdot\mathbf{w} = a_1a_2 + b_1b_2 + c_1c_2.$$

(12) $\mathbf{v}\cdot\mathbf{v} =$ ________ (12).

(13) Using the angle θ made by $\mathbf{u}$ and $\mathbf{v}$, we have $\mathbf{u}\cdot\mathbf{v} =$ ________ (13). The vectors $\mathbf{u}$ and $\mathbf{v}$ are *orthogonal* if ________ (14).

(14) The angle between $\mathbf{v}$ and $\mathbf{w}$ is ________ (15) if $\mathbf{v}\cdot\mathbf{w} > 0$ and ________ (16) if $\mathbf{v}\cdot\mathbf{w} < 0$.

(15) Assume $\mathbf{v} \neq 0$. Every vector $\mathbf{u}$ has a decomposition $\mathbf{u} = \mathbf{u}_{||\mathbf{v}} + \mathbf{u}_{\perp\mathbf{v}}$, where $\mathbf{u}_{||\mathbf{v}}$ is parallel to $\mathbf{v}$, and $\mathbf{u}_{\perp\mathbf{v}}$ is perpendicular to $\mathbf{v}$. Explicitly,

$$\mathbf{u}_{||\mathbf{v}} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|}\right)\mathbf{e}_{\mathbf{v}}, \quad \mathbf{u}_{\perp\mathbf{v}} = \mathbf{u} - \mathbf{u}_{||\mathbf{v}},$$

where

$$\mathbf{e}_{\mathbf{v}} = \frac{\mathbf{v}}{\|\mathbf{v}\|}.$$

(16) If $\mathbf{u}\cdot\mathbf{v} = \mathbf{u}\cdot\mathbf{w}$, is it true that $\mathbf{v} = \mathbf{w}$? ________ (17)

Problems

(1) Let $R = (-2, 7)$. Calculate the following:

(a) The length of $\overrightarrow{OR}$.

(b) The components of $\mathbf{u} = \overrightarrow{PR}$, where $P = (1, 2)$.

(c) The point P such that $\overrightarrow{PR}$ has components $\langle -2, 7 \rangle$.

(2) Find the given vector:

(a) Unit vector $\mathbf{e_v}$ where $\mathbf{v} = \langle 3, 4 \rangle$.

(b) Vector of length 4 in the direction of of $\mathbf{u} = \langle -1, 1 \rangle$.

(c) Vector $\mathbf{v}$ of length 2 making an angle of $30°$ with the x-axis.

(3) Determine whether or not the two vectors are parallel:

(a) $\mathbf{u} = \langle 1, -2, 5 \rangle$, $\mathbf{v} = \langle -2, 4, -10 \rangle$.

(b) $\mathbf{u} = \langle 4, 2, -6 \rangle$, $\mathbf{v} = \langle 2, 1, 3 \rangle$.

(4) Find a vector parametrization for the line with the given description:

(a) Passes through $P = (1, 2, -8)$, direction vector $\mathbf{v} = \langle 2, 1, 3 \rangle$.

(b) Passes through $(-2, 0, -2)$ and $(4, 3, 7)$.

(c) Passes through $(1, 1, 1)$ parallel to the line through $(2, 0, -1)$ and $(4, 1, 3)$.

(5) Show that the lines $\mathbf{r}_1(t) = \langle -1, 2, 2 \rangle + t\langle 4, -2, 1 \rangle$ and $\mathbf{r}_2(s) = \langle 0, 1, 1 \rangle + s\langle 2, 0, 1 \rangle$ do not intersect.

(6) Find the intersection of the lines $\mathbf{r}_1(t) = \langle -1, 1 \rangle + t\langle 2, 4 \rangle$ and $\mathbf{r}_2(s) = \langle 2, 1 \rangle + s\langle -1, 6 \rangle$ in the plane.

(7) Find all values of b for which the vectors are orthogonal.

(a) $\langle b, 3, 2 \rangle, \quad \langle 1, b, 1 \rangle$.

(b) $\langle 4, -2, 7 \rangle, \quad \langle b^2, b, 0 \rangle$.

(8) Find the angle between $\mathbf{v}$ and $\mathbf{w}$ if $\mathbf{v} \cdot \mathbf{w} = \frac{1}{2}\|\mathbf{v}\|\|\mathbf{w}\|$.

(9) If $\mathbf{e}$ and $\mathbf{f}$ are unit vectors and $\|\mathbf{e} + \mathbf{f}\| = \frac{3}{2}$, compute $\|\mathbf{e} - \mathbf{f}\|$.

(10) Find the projection of $\mathbf{u}$ along $\mathbf{v}$.

(a) $\mathbf{u} = \langle -1, 2, 0 \rangle, \quad \mathbf{v} = \langle 2, 0, 1 \rangle$.

(b) $\mathbf{u} = 5\mathbf{i} + 7\mathbf{j} - 4\mathbf{k}, \quad \mathbf{v} = \mathbf{k}$.

(11) Find the decomposition $\mathbf{a} = \mathbf{a}_{||\mathbf{b}} + \mathbf{a}_{\perp \mathbf{b}}$.

(a) $\mathbf{a} = \langle 4, -1, 5 \rangle, \quad \mathbf{b} = \langle 2, 1, 1 \rangle$.

(b) $\mathbf{a} = \langle x, y \rangle, \quad \mathbf{b} = \langle 1, -1 \rangle$.