Arc length, speed, and more

Name: Solutions

Math 1920 - Sections 221 and 222 - TA: Itamar Oliveira

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Review

(1) The length s of a path $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ for $a \leq t \leq b$ is

$$s = \int_a^b \|\mathbf{r}'(t)\|dt = \int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}dt.$$

(2) Arc length function: $s(t) = \boxed{\int_a^t \|\mathbf{r}'(u)\|du}$ (1) .

(3) *Speed* is the derivative of distance traveled with respect to time:

$$v(t) = \frac{ds}{dt} = \|\mathbf{r}'(t)\|.$$

(4) $\mathbf{r}(s)$ is an *arc length parametrization* if $\|\mathbf{r}'(s)\| = 1$ for all s.

(5) If $\mathbf{r}(t)$ is any parametrization such that $\mathbf{r}'(t) \neq 0$ for all t, then

$$\mathbf{r}_1(s) = \mathbf{r}(g^{-1}(s))$$

is an arc length parametrization, where $t = g^{-1}(s)$ is the inverse function of the arc length function $s = g(t)$.

Problems

(1) Compute the arc length of the following curves over the given interval:

(a) $\mathbf{r}(t) = \langle \cos t, \sin t, t^{3/2} \rangle$, $0 \le t \le 2\pi$.
Solution: $\frac{2}{27}((2+9\pi)^{3/2}\sqrt{2} - 4)$.

(b) $\mathbf{r}(t) = \langle t, 4t^{3/2}, 2t^{3/2} \rangle$, $0 \le t \le 3$.
Solution: $\frac{2}{135}(136^{3/2} - 1)$.

(2) Find the speed of $\mathbf{r}(t) = \langle t, \ln t, (\ln t)^2 \rangle$ at $t = 1$.
Solution: $\sqrt{2}$.

(3) Find an arc length parametrization of the circle in the plane $z = 9$ with radius 4 and center $(1, 4, 9)$.
Solution: $\mathbf{r}_1(s) = \langle 1 + 4\cos(s/4), 4 + 4\sin(s/4), 9 \rangle$.

(4) Let $\mathbf{r}(t) = \langle 3\cos t, 5\sin t, 4\cos t \rangle$. Show that $\|\mathbf{r}(t)\|$ is constant and use this to conclude that $\mathbf{r}(t)$ and $\mathbf{r}'(t)$ are orthogonal.

(5) Find the solution to the following differential equation with the given initial conditions: $\mathbf{r}''(t) = \langle e^t, \sin t, \cos t \rangle$, $\mathbf{r}(0) = \langle 1, 0, 1 \rangle$, $\mathbf{r}'(0) = \langle 0, 2, 2 \rangle$.
Solution: $\mathbf{r}(t) = \langle e^t - t, -\sin t + 3t, -\cos t + 2t + 2 \rangle$.