## Review

(1) If $f$ is a function of $x, y, z$ and if $x, y, z$ depend on two other variables, say, $s$ and $t$, then

$$
f(x, y, z)=f(x(s, t), y(s, t), z(s, t))
$$

is the composite function of $s$ and $t$. We refer to $s$ and $t$ as the independent variables.
(2) The Chain Rule expresses the partial derivatives with respect to the independent variables:

$$
\begin{aligned}
& \frac{\partial f}{\partial s}=\frac{\partial f}{\partial x} \frac{\partial x}{\partial s}+\frac{\partial f}{\partial y} \frac{\partial y}{\partial s}+\frac{\partial f}{\partial z} \frac{\partial z}{\partial s} \\
& \frac{\partial f}{\partial t}=\frac{\partial f}{\partial x} \frac{\partial x}{\partial t}+\frac{\partial f}{\partial y} \frac{\partial y}{\partial t}+\frac{\partial f}{\partial z} \frac{\partial z}{\partial t}
\end{aligned}
$$

(3) The Chain Rule (for a function $f$ in $n$ variables) can be expressed as a dot product:

$$
\frac{\partial f}{\partial t_{k}}=\left\langle\frac{\partial f}{\partial x_{1}}, \frac{\partial f}{\partial x_{2}}, \ldots, \frac{\partial f}{\partial x_{n}}\right\rangle \cdot\left\langle\frac{\partial x_{1}}{\partial t_{k}}, \frac{\partial x_{2}}{\partial t_{k}}, \ldots, \frac{\partial x_{n}}{\partial t_{k}}\right\rangle
$$

(4) Implicit differentiation is used to find the partial derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ where $z$ is defined implicitly by an equation $F(x, y, z)=0$ :

$$
\begin{aligned}
& \frac{\partial z}{\partial x}=-\frac{F_{x}}{F_{z}} \\
& \frac{\partial z}{\partial y}=-\frac{F_{y}}{F_{z}}
\end{aligned}
$$

(5) We say that a point $P=(a, b)$ is a critical point of $f(x, y)$ if $f_{x}(a, b)=0$ or does not exist, and $f_{y}(a, b)=0$ or does not exist.
(6) The local minimum or maximum values of $f$ occur at critical points.
(7) The discriminant of $f(x, y)$ at $P=(a, b)$ is

$$
D(a, b)=f_{x x}(a, b) f_{y y}(a, b)-f_{x y}^{2}(a, b)
$$

(8) Second Derivative Test: If $P=(a, b)$ is a critical point of $f(x, y)$, then

$$
\begin{gathered}
D(a, b)>0, \quad f_{x x}(a, b)>0 \Rightarrow f(a, b) \text { is a local miminum } \\
D(a, b)>0, \quad f_{x x}(a, b)<0 \Rightarrow f(a, b) \text { is a local maximum } \\
D(a, b)<0 \Rightarrow \text { saddle point } \\
D(a, b)=0 \Rightarrow \text { test inconclusive }
\end{gathered}
$$

(9) If $f$ is continuous on a closed and bounded domain $\mathcal{D}$, then $f$ takes on both a minimum and maximum value on $\mathcal{D}$. The extreme values occur either at critical points in the interior of $\mathcal{D}$ or at points on the boundary of $\mathcal{D}$.

## Problems

(1) Use the Chain Rule to calculate the partial derivatives. Express the answer in terms of the independent variables.
(a) $\frac{\partial h}{\partial t_{2}} ; h(x, y)=\frac{x}{y}, x=t_{1} t_{2}, y=t_{1}^{2} t_{2}$.
(b) $\frac{\partial F}{\partial y} ; F(u, v)=e^{u+v}, u=x^{2}, v=x y$.
(2) Suppose that $z$ is defined implicitly as a function of $x$ and $y$ by the equation $F(x, y, z)=x z^{2}+y^{2} z+$ $x y-1=0$.
(a) Calculate $F_{x}, F_{y}, F_{z}$.
(b) Calculate $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.
(3) Find the critical points of the following functions, then use the Second Derivative test to determine whether they are local minima, local maxima, or saddle points (or state that the test fails).
(a) $f(x, y)=4 x-3 x^{3}-2 x y^{2}$.
(b) $f(x, y)=\ln (x)+2 \ln (y)-x-4 y$.
(4) Find the maximum of $f(x, y)=y^{2}+x y-x^{2}$ on the square $0 \leq x \leq 2,0 \leq y \leq 2$.

