Optimization in several variables

Math 1920 - Sections 221 and 222 - TA: Itamar Oliveira

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Review

(1) We say that a point $P = (a, b)$ is a *critical point* of $f(x, y)$ if $f_x(a, b) = 0$ or does not exist, and $f_y(a, b) = 0$ or does not exist.

(2) The local minimum or maximum values of f occur at critical points.

(3) The *discriminant* of $f(x, y)$ at $P = (a, b)$ is

$$D(a, b) = f_{xx}(a, b)f_{yy}(a, b) - f_{xy}^2(a, b).$$

(4) *Second Derivative Test:* If $P = (a, b)$ is a critical point of $f(x, y)$, then

$$D(a, b) > 0, \quad f_{xx}(a, b) > 0 \Rightarrow \text{ } f(a, b) \text{ is a local miminum}$$
$$D(a, b) > 0, \quad f_{xx}(a, b) < 0 \Rightarrow \text{ } f(a, b) \text{ is a local maximum}$$
$$D(a, b) < 0 \Rightarrow \text{ saddle point}$$
$$D(a, b) = 0 \Rightarrow \text{ test inconclusive}$$

(5) If f is continuous on a closed and bounded domain $\mathcal{D}$, then f takes on both a minimum and maximum value on $\mathcal{D}$. The extreme values occur either at critical points in the interior of $\mathcal{D}$ or at points on the boundary of $\mathcal{D}$.

Problems

(1) Find the critical points of the following functions, then use the Second Derivative test to determine whether they are local minima, local maxima, or saddle points (or state that the test fails).

(a) $f(x, y) = x^4 + y^4 - 4xy$.
Solution: $(0, 0)$ is a saddle point, $f(1, 1)$ and $f(-1, -1)$ are local minima.

(b) $f(x, y) = x - y^2 - \ln(x + y)$.
Solution: $(\frac{3}{2}, -\frac{1}{2})$ is a saddle point.

(2) Find the maximum of $f(x, y) = y^2 + xy - x^2$ on the square $0 \leq x \leq 2$, $0 \leq y \leq 2$.

Solution: $f(1, 2) = 5$.

(3) Determine the global extreme values of the function on the given domain.

(a) $f(x, y) = x^3 - 2y; \quad 0 \leq x \leq 1, 0 \leq y \leq 1$.
Solution: The smallest value is $f(0, 1) = -2$ and it is the global minimum of f on the square. The global maximum is the largest value $f(1, 0) = 1$.

(b) $f(x, y) = x^3 + x^2y + 2y^2; \quad x, y \geq 0, x + y \leq 1$.
Solution: The global minimum of f in the given domain is $f(0, 0) = 0$ and the global maximum is $f(0, 1) = 2$.

(c) $f(x, y) = 2xy - x - y; \quad y \leq 4, y \geq x^2$.
Solution: The global maximum is $f(2, 4) = 10$ and the global minimum is $f(-2, 4) = -18$.

(4) Find three positive numbers that sum to 150 with the largest possible product of the three.

Solution: The three numbers are 50.