Change of Variables and Applications — Name: Solutions

Math 1920 - Sections 221 and 222 - TA: Itamar Oliveira — October 23, 2018

Review

(1) Total mass of regions $\mathcal{D} \subset \mathbb{R}^2$ and $\mathcal{W} \subset \mathbb{R}^3$:

$$M = \iint_{\mathcal{D}} \delta(x,y)\mathrm{d}A, \quad M = \iiint_{\mathcal{W}} \delta(x,y,z)\mathrm{d}V,$$

where δ is the mass density. The same formula can be used to compute the total charge of a region if δ is the charge density.

(2) Moments:

$$\underline{\text{In } \mathbb{R}^2:} \quad M_x = \iint_{\mathcal{D}} y\delta(x,y)\mathrm{d}A, \quad M_y = \iint_{\mathcal{D}} x\delta(x,y)\mathrm{d}A$$

$$\underline{\text{In } \mathbb{R}^3:} \quad M_{yz} = \iiint_{\mathcal{W}} x\delta(x,y,z)\mathrm{d}V, \quad M_{xz} = \iiint_{\mathcal{W}} y\delta(x,y,z)\mathrm{d}V, \quad M_{xy} = \iiint_{\mathcal{W}} z\delta(x,y,z)\mathrm{d}V.$$

(3) Coordinates of the *center of mass*:

$$\underline{\text{In } \mathbb{R}^2:} \quad x_{CM} = \frac{M_y}{M}, \quad y_{CM} = \frac{M_x}{M}.$$

$$\underline{\text{In } \mathbb{R}^3:} \quad x_{CM} = \frac{M_{yz}}{M}, \quad y_{CM} = \frac{M_{xz}}{M} \quad z_{CM} = \frac{M_{xy}}{M}.$$

(4) Random variables X and Y have joint probability density function $p(x,y)$ if

$$P(a \leq X \leq b; c \leq Y \leq d) = \int_{x=a}^{b} \int_{y=c}^{d} p(x,y)\mathrm{d}y\mathrm{d}x.$$

(5) A joint probability density function must satisfy $p(x,y) \geq 0$ and

$$\int_{x=-\infty}^{\infty} \int_{y=-\infty}^{\infty} p(x,y)\mathrm{d}y\mathrm{d}x = 1.$$

(6) Let $G(u,v) = (x(u,v), y(u,v))$ be a mapping. The *Jacobian* of G is the determinant

$$\mathrm{Jac}(G) = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\[2ex] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}.$$

(7) $\mathrm{Jac}(G) = [\mathrm{Jac}(F)]^{-1}$, where $F = G^{-1}$.

(8) Change of Variables Formula: If $G : \mathcal{D}_0 \to \mathcal{D}$ has component functions with continuous partial derivatives and one-to-one on the interior of $\mathcal{D}_0$, and if f is continuous, then

$$\iint_{\mathcal{D}} f(x,y)\mathrm{d}x\mathrm{d}y = \iint_{\mathcal{D}_0} f(x(u,v), y(u,v))\frac{\partial(u,v)}{\partial(x,y)}\mathrm{d}u\mathrm{d}v.$$

Problems

(1) Compute the total mass of the plate in Figure 1 assuming a mass density of $f(x,y) = \frac{x^2}{(x^2+y^2)}$ g/cm^2.

Solution: $\frac{50}{3}\pi + 25\sqrt{3}$.

(2) Find the z_{CM} (z coordinate of the center of mass) of a cylinder of radius 2 and height 4 and mass density e^{-z}, where z is the height above the base (an expression in terms of e is fine). Can you find the x_{CM} and y_{CM} without explicitly computing any integral?

Solution: $x_{CM} = y_{CM} = 0$, $z_{CM} = \frac{(1-5e^{-4})}{1-e^{-4}}$.

(3) Find a constant C such that

$$p(x,y) = \begin{cases} Cxy & \text{if } 0 \leq x \text{ and } 0 \leq y \leq 1-x \\ 0 & \text{otherwise} \end{cases}$$

is a joint probability density function, then calculate $P(X \geq Y)$.

Solution: $C = 24$, $P(X \geq Y) = \frac{1}{2}$.

(4) Compute $\iint_{\mathcal{D}} (x+3y)\mathrm{d}x\mathrm{d}y$, where $\mathcal{D}$ is the shaded region in Figure 3. *Hint:* Use the map $\Phi(u,v) = (u-2v, v)$.

Solution: 80.

(5) Find a mapping Φ that maps the disk $u^2+v^2 \leq 1$ onto the interior of the ellipse $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 \leq 1$. Then use the Change of Variables Formula to prove that the area of the ellipse is πab.

Solution: The map $\Phi(u,v) = (au, bv)$ does the job.

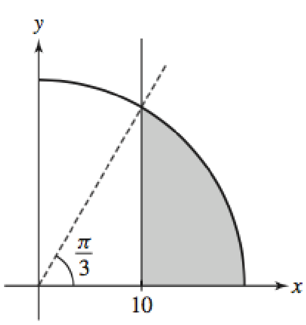

Figure 1: Problem 1

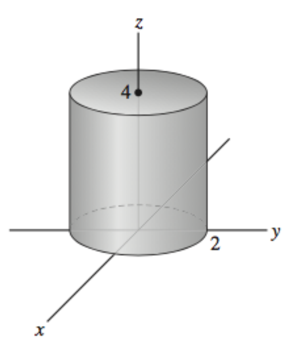

Figure 2: Problem 2

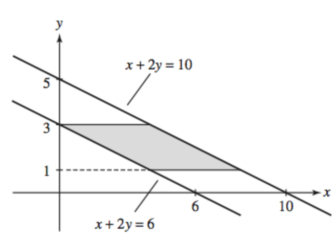

Figure 3: Problem 4