Conservative Vector Fields

Math 1920 - Sections 221 and 222 - TA: Itamar Oliveira

Name: Solutions

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Review

(1) A vector field $\mathbf{F}$ on a domain $\mathcal{D}$ is called *path-independent* if for any two points $P, Q \in \mathcal{D}$, we have

$$\int_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{r} = \int_{\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{r}$$

for any two paths $\mathcal{C}_1$ and C_2 in $\mathcal{D}$ from P to Q.

(2) The Fundamental Theorem for Conservative Vector Fields: If $\mathbf{F} = \nabla f$, then

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = f(Q) - f(P)$$

for any path $\mathbf{r}$ from P to Q in the domain of $\mathbf{F}$. This shows that conservative vector fields are path independent. The converse is also true: on an open, connected domain, a path-independent vector field is conservative.

(3) The work W exerted on an object along a curve $\mathcal{C}$ is given by:

$$W = \int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}.$$

The work performed *against* $\mathbf{F}$ is the quantity $-\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$.

Problems

(1) Evaluate

$$\oint_{\mathcal{C}} \sin x\,dx + z\cos y\,dy + \sin y\,dz$$

where $\mathcal{C}$ is the ellipse $4x^2 + 9y^2 = 36$, oriented clockwise.

Solution: This vector field is conservative, with potential function $f(x, y, z) = z\sin y - \cos x$. This way, the integral is 0.

(2) Determine whether the vector field in the picture below is conservative or not. Solution: The line integral from the lower left corner to the upper right corner would clearly be larger if the path passed through the lower right region than if it passed through the upper left region, therefore the vector field is not conservative.

(3) Let $\mathbf{F}(x, y) = \left\langle \frac{1}{x}, \frac{-1}{y} \right\rangle$. Show that the work against $\mathbf{F}$ required to move an object from $(1, 1)$ to $(3, 4)$ is the same no matter what path we pick in the first quadrant. Compute this work.

Solution: This vector field is conservative, hence the work does not depend on the path in the first quadrant. A potential function is $f(x, y) = \ln y - \ln x$.

$$W\ \textbf{(against F)} = \ln 4 - \ln 3.$$

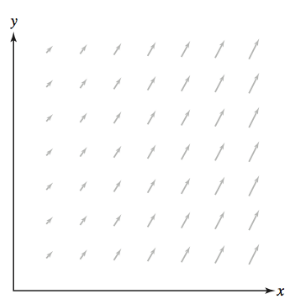

Figure 1: Problem 2