More on Conservative Vector Fields   Name: ____________

Math 1920 - Sections 221 and 222 - TA: Itamar Oliveira   November 8, 2018

Review

(1) A vector field $\mathbf{F}$ on a domain $\mathcal{D}$ is called *path-independent* if for any two points $P, Q \in \mathcal{D}$, we have

$$\int_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{r} = \int_{\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{r}$$

for any two paths $\mathcal{C}_1$ and C_2 in $\mathcal{D}$ from P to Q.

(2) The Fundamental Theorem for Conservative Vector Fields: If $\mathbf{F} = \nabla f$, then

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = f(Q) - f(P)$$

for any path $\mathbf{r}$ from P to Q in the domain of $\mathbf{F}$. This shows that conservative vector fields are path independent. The converse is also true: on an open, connected domain, a path-independent vector field is conservative.

(3) The work W exerted on an object along a curve $\mathcal{C}$ is given by:

$$W = \int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}.$$

The work performed *against* $\mathbf{F}$ is the quantity $-\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$.

Problems

(1) Calculate the work required to move an object from $P = (1, 1, 1)$ to $Q = (3, -4, -2)$ against the force field $\mathbf{F}(x, y, z) = -12r^{-4}\langle x, y, z\rangle$, where $r = \sqrt{x^2 + y^2 + z^2}$.

(2) Let $\mathbf{F}(x, y) = \langle 9y - y^3, e^{\sqrt{y}}(x^2 - 3x)\rangle$, and let $\mathcal{C}_2$ be the oriented curve in the picture below.

(a) Show that $\mathbf{F}$ is not conservative.

(b) Show that $\int_{\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{r} = 0$ without explicitly computing this integral. *Hint:* Show that $\mathbf{F}$ is orthogonal to the edges along the square.

(3) Find a conservative vector field of the form $\mathbf{F} = \langle g(y), h(x)\rangle$ such that $\mathbf{F}(0, 0) = \langle 1, 1\rangle$, where $g(y)$ and $h(x)$ are differentiable functions. Determine all such vector fields.

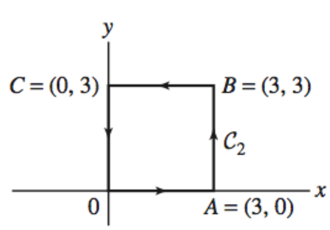

Figure 1: Problem 2