Surface Integrals Review

Math 1920 - Sections 221 and 222 - TA: Itamar Oliveira

Name: Solutions

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Review

(1) A *parametrized surface* is a surface $\mathcal{S}$ whose points are described in the form

$$G(u,v) = (x(u,v), y(u,v), z(u,v)),$$

where the *parameters* u and v vary in a domain $\mathcal{D}$ in the uv-place.

(2) Tangent and normal vectors:

- Tangent vectors:

$$\mathbf{T}_u = \frac{\partial G}{\partial u} = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle, \quad \mathbf{T}_v = \frac{\partial G}{\partial v} = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right\rangle.$$

- Normal vector:

$$\mathbf{N} = \mathbf{N}(u,v) = \mathbf{T}_u \times \mathbf{T}_v.$$

(3) The quantity $\|\mathbf{N}\|$ is an "area distortion factor". If $\mathcal{D}$ is a small region in the uv-plane and $\mathcal{S} = G(\mathcal{D})$, then

$$\text{Area}(\mathcal{S}) \approx \|\mathbf{N}(u_0, v_0)\| \text{Area}(\mathcal{D}),$$

where (u_0, v_0) is *regular* at (u,v) if $\mathbf{N}(u,v) \neq \mathbf{0}$.

(4) Surface integrals and surface area:

$$\iint_{\mathcal{S}} f(x,y,z) dS = \iint_{D} f(G(u,v)) \|\mathbf{N}(u,v)\| \, du \, dv,$$

$$\text{Area}(\mathcal{S}) = \iint_{\mathcal{D}} \|\mathbf{N}(u,v)\| \, du \, dv.$$

(5) Some standard parametrizations:

- Cylinder of radius R (z-axis as central axis):

$$G(\theta, z) = (R\cos\theta, R\sin\theta, z), \quad \mathbf{N} = \mathbf{T}_\theta \times \mathbf{T}_z = R\langle \cos\theta, \sin\theta, 0\rangle, \quad dS = \|\mathbf{N}\| \, d\theta \, dz = R \, d\theta \, dz.$$

- Sphere of radius R, centered at the origin:

$$G(\theta, z) = (R\cos\theta\sin\phi, R\sin\theta\sin\phi, R\cos\phi), \quad \text{Unit radial vector:} \quad \mathbf{e}_r \langle \cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi\rangle,$$

$$\text{Outward normal:} \quad \mathbf{N} = \mathbf{T}_\phi \times \mathbf{T}_\theta = (R^2 \sin\phi) \, \mathbf{e}_r, \quad dS = \|\mathbf{N}\| \, d\phi \, d\theta = R^2 \sin\phi \, d\phi \, d\theta.$$

(6) Graph of $z = g(x,y)$:

$$G(x,y) = (x, y, g(x,y)), \quad \mathbf{N} = \mathbf{T}_x \times \mathbf{T}_y = \langle -g_x, -g_y, 1\rangle, \quad dS = \|\mathbf{N}\| \, dx \, dy = \sqrt{1 + g_x^2 + g_y^2} \, dx \, dy.$$

Problems

(1) Let $\mathcal{S} = G(\mathcal{D})$, where $\mathcal{D} = \{(u,v) : u^2 + v^2 \leq 1, u \geq 0, v \geq 0\}$ and $G(u,v) = (2u+1, u-v, 3u+v)$.

(a) Calculate the surface area of $\mathcal{S}$. Solution: $\frac{\pi\sqrt{6}}{2}$.

(b) Evaluate $\iint_{\mathcal{S}} (x-y)dS$. *Hint: use polar coordinates.* Solution: Integrate over $0 \leq r \leq 1$ and $0 \leq \theta \leq \frac{\pi}{2}$. The result is $\frac{8+3\pi}{\sqrt{6}}$.

(2) Calculate $\iint_{\mathcal{S}} f(x,y,z)dS$ for the given surface and function.

(a) $G(u,v) = (u, v^3, u+v)$, $0 \leq u \leq 1$, $0 \leq v \leq 1$; $f(x,y,z) = y$. Solution: $\frac{19\sqrt{19}-1}{108}$.

(b) Part of the unit sphere centered at the origin, where $x \geq 0$ and $|y| \leq x$; $f(x,y,z) = x$. Solution: $\frac{\pi}{\sqrt{2}}$.

(3) Find the surface area of the part of the cone $x^2 + y^2 = z^2$ between the planes $z = 2$ and $z = 5$. Solution: Parametrize it by $\Phi(u,v) = (u\cos v, u\sin v, u)$, $\mathcal{D} : 0 \leq v \leq 2\pi, 2 \leq u \leq 5$. The normal vector $\mathbf{N}$ will be $\mathbf{N}(u,v) = \langle -u\cos v, -u\sin v, u\rangle$. Use that to conclude that the area is $21\sqrt{2}\pi$.

(4) Calculate $\iint_G x^2 z dS$, where G is the cylinder (including the top and the bottom) $x^2+y^2 = 4$, $0 \leq z \leq 3$. Solution: Compute the top, bottom and side contributions separately and add them up. The answer is 48π.