Surface Integrals and Green's Theorem   Name: Solutions

Math 1920 - Sections 221 and 222 - TA: Itamar Oliveira   November 20, 2018

Review

(1) A surface $\mathcal{S}$ is *oriented* if a continuously varying unit normal vector $\mathbf{n}(P)$ is specified at each point on $\mathcal{S}$. This distinguishes an "outward" direction on the surface.

(2) The integral of a vector field $\mathbf{F}$ over an oriented surface $\mathcal{S}$ is defined as the integral of the normal component $\mathbf{F} \cdot \mathbf{n}$ over $\mathcal{S}$.

(3) Vector surface integrals are computed using the formula

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} = \iint_{\mathcal{D}} \mathbf{F}(G(u,v)) \cdot \mathbf{N}(u,v)\, du\, dv$$

Here, $G(u,v)$ is a parametrization of $\mathcal{S}$ such that $\mathbf{N}(u,v) = \mathbf{T}_u \times \mathbf{T}_v$ points in the direction of the unit normal vector specified by the orientation.

(4) The surface integral of a vector field $\mathbf{F}$ over $\mathcal{S}$ is also called the *flux* of $\mathbf{F}$ through G. If $\mathbf{F}$ is the velocity field of a fluid, then $\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S}$ is the rate at which fluid flows through $\mathcal{S}$ per unit time.

(5) We have two notations for the line integral of a vector field on the plane:

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} \quad \text{and} \quad \int_{\mathcal{C}} F_1\, dx + F_2\, dy.$$

(6) $\partial\mathcal{D}$ denotes the boundary of $\mathcal{D}$ with its boundary orientation.

(7) Green's Theorem:

$$\oint_{\partial\mathcal{D}} F_1\, dx + F_2\, dy = \iint_{\mathcal{D}} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA$$

or

$$\oint_{\partial\mathcal{D}} \mathbf{F} \cdot d\mathbf{r} = \iint_{\mathcal{D}} \text{curl}_z(\mathbf{F})\, dA.$$

(8) Formulas for the area of a region $\mathcal{D}$ enclosed by $\mathcal{C}$:

$$\text{Area}(\mathcal{D}) = \oint_{\mathcal{C}} x\, dy = \oint_{\mathcal{C}} -y\, dx = \frac{1}{2} \oint_{\mathcal{C}} x\, dy - y\, dx.$$

(9) The quantity

$$\text{curl}_z(\mathbf{F}) = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$$

is interpreted as *circulation per unit area*. If $\mathcal{D}$ is a small domain with boundary $\mathcal{C}$, then for any $P \in \mathcal{D}$,

$$\oint_{\mathcal{C}} F_1\, dx + F_2\, dy \approx \text{curl}_z(\mathbf{F})(P) \cdot \text{Area}(\mathcal{D}).$$

(10) Vector Form of Green's Theorem:

$$\oint_{\partial\mathcal{D}} \mathbf{F} \cdot \mathbf{n}\, ds = \iint_{\mathcal{D}} \text{div}(\mathbf{F})\, dA.$$

The right-hand side of the identity above is called the *flux* of $\mathbf{F}$ out of the unit circle.

Problems

(1) Calculate $\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S}$ for the given surface and vector field.

(a) $\mathbf{F} = \langle e^z, z, x \rangle$, $G(r,s) = (rs, r+s, r)$, $0 \le r \le 1$, $0 \le s \le 1$, oriented by $\mathbf{T}_r \times \mathbf{T}_s$.
Solution: $\frac{4}{3} - e$.

(b) $\mathbf{F} = \langle z, z, x \rangle$, $z = 9 - x^2 - y^2$, $x \ge 0$, $y \ge 0$, $z \ge 0$, upward-pointing normal.
Solution: $\frac{693}{5}$.

(2) Let $\mathcal{S}$ be the ellipsoid $\left(\frac{x}{4}\right)^2 + \left(\frac{y}{3}\right)^2 + \left(\frac{z}{2}\right)^2 = 1$. Calculate the flux of $\mathbf{F} = z\mathbf{i}$ over the portion of $\mathcal{S}$ where $x, y, z \le 0$ with upward-pointing normal.

Solution: Parametrize it by $\Phi(\theta, \phi) = (4\cos\theta\sin\phi, 3\sin\theta\sin\phi, 2\cos\phi)$, with the parameter domain $\mathcal{D} = \{(\theta,\phi) : \pi \le \theta \le \frac{3\pi}{2}, \quad \frac{\pi}{2} \le \phi < \pi\}$. Compute the tangent and normal vectors, the dot product $\mathbf{F} \cdot \mathbf{N}$ and use this to compute $\iint_S \mathbf{F} \cdot d\mathbf{S}$. The answer is -4.

(3) Use Green's Theorem to evaluate the line integral. Orient the curve counterclockwise.

(a) $\oint_{\mathcal{C}} y^2\, dx + x^2\, dy$, where $\mathcal{C}$ is the boundary of the unit square $0 \le x \le 1$, $0 \le y \le 1$.

Solution: 0.

(b) $\oint_{\mathcal{C}} e^{2x+y}\, dx + e^{-y}\, dy$, where $\mathcal{C}$ is the triangle with vertices $(0,0)$, $(1,0)$, and $(1,1)$.

Solution: $\frac{e^2}{2} - \frac{e^3}{3} - \frac{1}{6}$.

(4) Evaluate $I = \int_{\mathcal{C}} (\sin x + y)\, dx + (3x + y)\, dy$ for the nonclosed path $ABCD$ in Figure 1.
Solution: 34.

(5) Let $\mathcal{C}_R$ be the circle of radius R centered at the origin. Use the general form of Green's Theorem to determine $\oint_{\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}$ is a vector field such that $\oint_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{r} = 9$ and $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = x^2 + y^2$ for (x,y) in the annulus $1 \le x^2 + y^2 \le 4$.
Solution: $9 + \frac{15\pi}{2}$.

(6) Referring to Figure 2, suppose that

$$\oint_{\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{r} = 3\pi \quad \text{and} \quad \oint_{\mathcal{C}_3} \mathbf{F} \cdot d\mathbf{r} = 4\pi.$$

Use Green's Theorem to determine the circulation of $\mathbf{F}$ around $\mathcal{C}_1$, assuming that $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 9$ on the shaded region.

Solution: 214π.

(7) Compute the flux of $\mathbf{F}(x,y) = \langle xy^2 + 2x, x^2y - 2y \rangle$ across the simple closed curve that is the boundary of the half-disk given by $x^2 + y^2 \le 9$, $y \ge 0$.
Solution: $\frac{81\pi}{4}$.

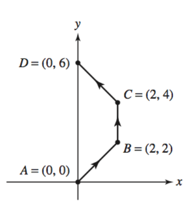

Figure 1: Problem 4.

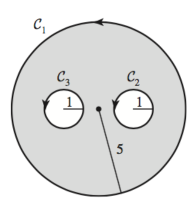

Figure 2: Problem 6.