Stokes' Theorem Review — Name: Solutions

Math 1920 - Sections 221 and 222 - TA: Itamar Oliveira — November 27, 2018

Review

(1) The *boundary* of a surface $\mathcal{S}$ is denoted by ∂S. We say that $\mathcal{S}$ is *closed* if ∂S is empty.

(2) Suppose that $\mathcal{S}$ is oriented (a continuously varying unit normal is specified at each point of $\mathcal{S}$). The *boundary orientation* of ∂S is defined as follows: If you walk along the boundary in the positive direction with your left hand pointing in the normal direction, then the surface is on your left.

(3) Stokes' Theorem relates the circulation around the boundary to the surface integral of the curl:

$$\oint_{\partial S} \mathbf{F} \cdot d\mathbf{r} = \iint_{\mathcal{S}} \operatorname{curl}(\mathbf{F}) \cdot d\mathbf{S}.$$

(4) Surface independence: If $\mathbf{F} = \operatorname{curl}(\mathbf{A})$, then the flux of $\mathbf{F}$ through a surface $\mathcal{S}$ depends only on the oriented boundary ∂S and not on the surface itself:

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} = \oint_{\partial S} \mathbf{A} \cdot d\mathbf{r}.$$

In particular, if $\mathcal{S}$ is *closed* (i.e., ∂S is empty) and $\mathbf{F} = \operatorname{curl}(\mathbf{A})$, then $\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} = 0$. If $\mathcal{S}_1$ and $\mathcal{S}_2$ are oriented surfaces that share an oriented boundary and $\mathbf{F} = \operatorname{curl}(\mathbf{A})$, then

$$\iint_{\mathcal{S}_1} \mathbf{F} \cdot d\mathbf{S} = \iint_{\mathcal{S}_2} \mathbf{F} \cdot d\mathbf{S}.$$

(5) The curl is interpreted as a vector that encodes circulation per unit area: If P is any point and $\mathbf{n}$ is a unit vector, then

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} \approx (\operatorname{curl}(\mathbf{F}) \cdot \mathbf{n}) \operatorname{Area}(\mathcal{D}),$$

where $\mathcal{C}$ is a small, simple closed curve around P in the plane through P with unit normal vector $\mathbf{n}$, and $\mathcal{D}$ is the region enclosed by $\mathcal{C}$.

Problems

(1) Verify Stokes' Theorem for $\mathbf{F} = \langle yz, 0, x\rangle$ and $\mathcal{S}$ is the portion of the plane $\frac{x}{2} + \frac{y}{3} + z = 1$ where $x, y, z \geq 0$.

Solution: Show that $\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = -1 = \iint_S \operatorname{curl}(\mathbf{F}) \cdot d\mathbf{S}$.

(2) Apply Stokes' Theorem to evaluate $\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$ by finding the flux of curl($\mathbf{F}$) across an appropriate surface.

(a) $\mathbf{F} = \langle yz, xy, xz\rangle$, $\mathcal{C}$ is the square with vertices $(0, 0, 2)$, $(1, 0, 2)$, $(1, 1, 2)$, and $(0, 1, 2)$, oriented counterclockwise as viewed from above.

Solution: $-\frac{3}{2}$.

(b) $\mathbf{F} = \langle y, z, x\rangle$, $\mathcal{C}$ is the triangle with vertices $(0, 0, 0)$, $(3, 0, 0)$, and $(0, 3, 3)$, oriented counterclockwise as viewed from above.

Solution: 0.

(3) Let I be the flux of $\mathbf{F} = \langle e^y, 2xe^{x^2}, z^2\rangle$ through the upper hemisphere $\mathcal{S}$ of the unit sphere.

(a) Let $\mathbf{G} = \langle e^y, 2xe^{x^2}, 0\rangle$. Find a vector field $\mathbf{A}$ such that curl($\mathbf{A}$) = $\mathbf{G}$.

Solution: $\mathbf{A} = \langle 0, 0, e^y - e^{x^2}\rangle$.

(b) Use Stokes' theorem to show that the flux of $\mathbf{G}$ through $\mathcal{S}$ is zero. *Hint:* Calculate the circulation of $\mathbf{A}$ around ∂S.

Solution: Use $\iint_S \mathbf{G} \cdot d\mathbf{S} = \oint_{\mathcal{C}} \mathbf{A} \cdot d\mathbf{r}$.

(c) Calculate I. *Hint:* Use (b) to show that I is equal to the flux of $\langle 0, 0, z^2\rangle$ through $\mathcal{S}$.

Solution: Use $\mathbf{F} = \operatorname{curl}(\mathbf{A}) + \langle 0, 0, z^2\rangle$.

(4) Let $\mathbf{F} = \langle y^2, x^2, z^2\rangle$. Show that

$$\int_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{r} = \int_{\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{r}$$

for any two closed curves lying on a cylinder whose central axis is the z-axis (figure below).

Solution: Apply Stokes' theorem for the part of the cylinder for which $\mathcal{C}_1$ and $\mathcal{C}_2$ are boundary curves.

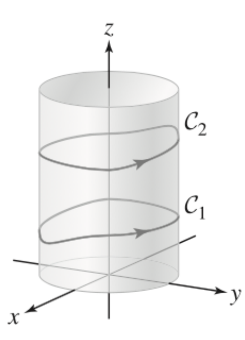

Figure 1: Problem 4.