Cellular structure on the minimal resolution of the edge ideal of the complement of the n-cycle

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Abstract: We study the minimal free resolution of the edge ideal of the complement of the n-cycle for $n \geq 4$ and construct a regular cellular complex which supports this resolution.

1 Introduction

Let $S = k[x_1, \ldots, x_n]$ be a polynomial ring in n variables over a field k. We are interested in the structure of the minimal free resolutions of quadratic monomial ideals of S. The method of polarization allows us to narrow our considerations to square-free quadratic monomial ideals. The minimal monomial generators of such an ideal can be easily encoded in a graph as follows: let G be a graph with vertex set $\{1, \ldots, n\}$, the *edge ideal of* G is the monomial ideal I_G of S whose minimal monomial generators are the monomials $x_i x_j$ where (i, j) is an edge of G. Much work has been done to discover connections between the combinatorial properties of the graph G and the algebraic properties of its edge ideal I_G. The properties of the complement graph G^c have turned out to be useful in this endeavour; recall that the complement of G is the graph G^c such that the vertex set of G^c the same as the vertex set of G and the edges of G^c are the non-edges of G. One of the main results about resolutions of edge ideals was proved by Fröberg [Frö88] and states that an edge ideal I_G has a linear minimal free resolution if and only if the complement graph G^c is chordal.

We consider the question of whether there exists a regular cellular structure which supports the minimal free resolution of an edge ideal. In [BW02], Batzies and Welker showed that if an edge ideal has a linear minimal free resolution then there is a CW-cellular complex which supports that resolution. Their proof is non-constructive, however. Corso and Nagel in [CN08] and [CN09] and Horwitz in [Hor07] construct explicit regular cellular structures for several classes of edge ideals with linear minimal free resolutions. In view of these results, we focus on edge ideals whose minimal free resolutions are not linear, but are close to being linear. The simplest non-chordal graphs are cycles of length four or greater and the simplest examples of edge ideals with non-linear resolutions are the edge ideals of the complements of such cycles. We study the minimal free resolutions of such ideals. By [EGHP05] and [FRG09] we know that the minimal free resolution of the edge ideal of the complement of the n-cycle is linear until homological degree $n-4$ and that the only non-zero Betti number in homological degree greater than $n-4$ is $\beta_{n-3,n} = 1$.

Let $I_n \subset S$ be the edge ideal of the complement of the n-cycle. That is, $I_n = (x_1x_3, x_1x_4, \ldots, x_1x_{n-1}, x_2x_4, x_2x_5, \ldots, x_2x_n, \ldots, x_{n-2}x_n)$. Let $J_n = I_n + (x_1x_n)$. We study the minimal free resolution of S/I_n by first considering the minimal free resolution of S/J_n. In Section 2 we construct an explicit resolution for S/J_n and a regular cellular complex which supports this resolution (a different cellular complex is constructed in [CN08]; see Remark 3.8). Then in Section 3 we obtain a regular cell complex which supports the resolution of S/I_n from that which we constructed for S/J_n.

2 The Resolution of S/J_n

We begin this section by constructing a basis and differential maps for the minimal free resolution of S/J_n. The minimal free resolution of S/J_n has basis $\{1\}$ in homological degree 0 and basis in homological degree $f+1$ the set of symbols $x = (x_ix_j; e_1, \ldots, e_c, e_{c+1} \ldots, e_{c+r}, \ldots, e_f)$ where x_ix_j is a minimal monomial generator of J_n with $i < j$, $e_1 < e_2 < \cdots < e_{c+r} < i < i+1 < e_{c+r+1} < \cdots < e_f < j$, and $e_{c+1} = i-r, e_{c+2} = i-r+1, \ldots, e_{c+r} = i-1, e_c \neq i-r-1$.

The differential is made up of three maps, ∂, μ_1, and μ_2 which we define below. First define $b(m)$ for a monomial m to be the largest (in the lex order with $x_1 > x_2 > \cdots > x_n$) minimal generator of the ideal J_n that divides m.

Then we define

$$\partial(x) = \sum_{p=1}^{f} \partial^{e_p}(x)$$

where

$$\partial^{e_p}(x) = (-1)^p x_{e_p}(x_ix_j; e_1, \ldots, \widehat{e}_p, \ldots e_f)\,.$$

The second map is defined by

$$\mu_1(x) = \sum_{q=1}^{f} \mu_1^{e_q}(x)$$

where

$$\mu_1^{e_q}(x) = (-1)^{q+1}\frac{x_ix_jx_q}{b(x_ix_jx_q)}(b(x_ix_jx_q); e_1, \ldots, \widehat{e}_q, \ldots, e_f)\,,$$

Finally, we define

$$\mu_2(x) = \sum_{s=c+1}^{c+r-1} \mu_2^{e_s}(x)$$

where

$$\mu_2^{e_s}(x) = (-1)^{c+r+1}x_{e_{s+1}}(x_{e_s}x_j; e_1, \ldots, e_c, e_{c+1}, \ldots, \widehat{e}_s, \widehat{e}_{s+1}, \ldots, e_{c+r}, i, e_{c+r+1}, \ldots, e_f)\,.$$

It will sometimes be the case that the symbols appearing in $\mu_1(x)$ are not valid elements the basis as defined above. It is understood in this case that those terms of $\mu_1(x)$ are zero.

Define $d(x) = \partial(x) + \mu_1(x) + \mu_2(x)$ for x in homological degree 2 or higher. In homological degree 1 define $d(x_ix_j; \emptyset) = x_ix_j$. Then $d(x)$ is the differential of the minimal free resolution of S/J_n with the basis described above. Before proving that the minimal free resolution of S/J_n has basis and differential as described, we prove the following lemma.

Lemma 2.1. *Let $x = (x_ix_j; e_1, \ldots, e_c, e_{c+1} \ldots, e_{c+r}, \ldots, e_f)$ as above. Then $d^2(x) = 0$.*

Proof. Every term of $d^2(x)$ has the form $(-1)^t x_u x_v \cdot y$ where y is the symbol for some basis element in homological degree f. We call $x_u x_v$ the *coefficient* of this term and we proceed by considering all the terms of $d^2(x)$ with the same coefficient $x_u x_v$ and show that these terms cancel.

First note that for all $p, q \in \{1, \ldots, f\}$ (assume without loss of generality that $p < q$), we have the following equality

$$\begin{aligned} \partial^{e_p} \circ \partial^{e_q}(x) &= (-1)^{p+q} x_{e_p} x_{e_q} (x_i x_j; e_1, \ldots, \widehat{e_p}, \ldots, \widehat{e_q}, \ldots, e_f) \\ &= -\partial^{e_q} \circ \partial^{e_p}(x)\,. \end{aligned}$$

In other words, $\partial^2(x) = 0$ for all x. In view of this, in the following we consider only the terms of $d^2(x)$ which do not come from $\partial^2(x)$.

There are several cases to consider, but first we make the following observations:

1. $b(x_i x_j x_{e_p}) = x_i x_{e_p}$ if $e_p \neq i - 1$
   $b(x_i x_j x_{e_p}) = x_j x_{e_p}$ if $e_p = i - 1$
   So μ_1 always contributes x_i or x_j to the coefficient of a term of $d(x)$

2. μ_2 always contributes x_{e_p} with $c + 2 \leq p \leq c + r$

3. ∂ always contributes x_{e_p} with $1 \leq p \leq f$.

Case 1. Consider the terms of $d^2(x)$ with the coefficient $x_{e_p} x_{e_q}$ where $p - 1, q - 1 \notin \{c+1, \ldots, c+r-1\}$, $p < q$. The only terms with this coefficient come from $\partial^{e_p} \circ \partial^{e_q}$ and $\partial^{e_q} \circ \partial^{e_p}$. We have already shown that $\partial^{e_p} \circ \partial^{e_q}(x) = -\partial^{e_q} \circ \partial^{e_p}(x)$, so we are done.

Case 2. Consider terms of $d^2(x)$ with the coefficient $x_{e_p} x_{e_q}$ where $p-1 \in \{c+1, \ldots, c+r-1\}, q-1 \notin \{c+1, \ldots, c+r-1\}$, again with $p < q$.

In this case μ_2 can also contribute to the coefficient $x_{e_p} x_{e_q}$ so we also have the terms

$$\begin{aligned} \partial^{e_q} \circ \mu_2^{e_{p-1}}(x) &= (-1)^{c+r+q} x_{e_p} x_{e_q} (x_{e_{p-1}} x_j; e_1, \ldots, e_{c+1} \ldots, \widehat{e_{p-1}}, \widehat{e_p}, \ldots, e_{c+r}, i, \ldots, \widehat{e_q}, \ldots, e_f) \\ &= -\mu_2^{e_{p-1}} \circ \partial^{e_q}(x)\,. \end{aligned}$$

The case where $p \in \{c+1, \ldots, c+r-1\}$ and $q \notin \{c+1, \ldots, c+r-1\}$ but $q < p$ is analogous and results in the same relation

$$\partial^{e_q} \circ \mu_2^{e_{p-1}}(x) = -\mu_2^{e_{p-1}} \circ \partial^{e_q}(x)\,.$$

Case 3. Next we consider terms of $d^2(x)$ with the coefficient $x_{e_p} x_{e_q}$ where $p - 1, q - 1 \in \{c+1, \ldots, c+r-1\}$ and $p < q$.

If $p < q - 1$, then as in Case 2 we have

$$\partial^{e_p} \circ \mu_2^{e_{q-1}}(x) = -\mu_2^{e_{q-1}} \circ \partial^{e_p}(x)\,.$$

In this case ($p < q - 1$), we also have the following relation

$$\begin{aligned} \partial^{e_q} \circ \mu_2^{e_{p-1}}(x) &= (-1)^{c+r+q-1} x_{e_q} x_{e_p} (x_{e_{p-1}} x_j; e_1, \ldots, e_{c+1} \ldots, \widehat{e_{p-1}}, \widehat{e_p}, \ldots, \widehat{e_q}, \ldots, e_{c+r}, i, \ldots, e_f) \\ &= -\mu_2^{e_{p-1}} \circ \mu_2^{e_{q-1}}(x)\,. \end{aligned}$$

Finally, if instead we have $p = q-1$, then

$$\begin{aligned} \mu_1^{q-2} \circ \mu_2^{q-1}(x) &= (-1)^{c+r+q} x_{e_q} x_{e_{q-1}} (x_{q-2} x_j; e_1, \dots, e_{c+1}, \dots, \widehat{e_{q-2}}, \widehat{e_{q-1}}, \widehat{e_q}, \dots, e_{c+r}, i, \dots, e_f) \\ &= -\partial^{e_q} \circ \mu_2^{q-2}(x)\,. \end{aligned}$$

Case 4. Consider the terms of $d^2(x)$ with the coefficient $x_i x_{e_p}$, $p \in \{1, \dots, f\}$. The only terms of $d^2(x)$ with x_i in the coefficient come from μ_1^{i-1} or from ∂^i.

If $p \neq c+r$ (recall that $e_{c+r} = i-1$), then we have

$$\partial^{e_p} \circ \mu_1^{i-1}(x) = -\mu_1^{i-1} \circ \partial^{e_p}(x)\,.$$

To see this in the case where $p < c+r$, note that

$$\begin{aligned} \partial^{e_p} \circ \mu_1^{i-1}(x) &= (-1)^{c+r+1+p} x_i x_{e_p} (x_{i-1} x_j; e_1, \dots, \widehat{e_p} \dots, \widehat{e_{c+r}}, \dots, e_f) \\ &= -\mu_1^{i-1} \circ \partial^{e_p}(x)\,. \end{aligned}$$

On the other hand, if $p = i-1$

$$\begin{aligned} \mu_1^{i-2} \circ \mu_1^{i-1}(x) &= (-1)^{c+r+1+c+r} x_i x_{i-1} (x_{i-2} x_j; e_1, \dots, e_c, e_{c+1}, \dots, \widehat{e_{c+r-1}}, \widehat{e_{c+r}}, \dots, e_f) \\ &= -\partial^i \circ \mu_2^{i-2}(x)\,. \end{aligned}$$

Finally, if $p-1 \in \{c+1, \dots, c+r-2\}$, then we also have the relation

$$\begin{aligned} \partial^i \circ \mu_2^{e_{p-1}}(x) &= (-1)^{c+r+c+r} x_i x_{e_p} (x_{e_{p-1}} x_j; e_1, \dots, e_c, e_{c+1}, \dots, \widehat{e_{p-1}}, \widehat{e_p}, \dots, e_{c+r}, \dots, e_f)) \\ &= -\mu_2^{e_{p-1}} \circ \mu_1^{i-1}(x)\,. \end{aligned}$$

Case 5. Now we consider terms of $d^2(x)$ with the coefficient $x_j x_{e_p}$, $p \in \{1, \dots, f\}$. There are two ways that x_j can be part of the coefficient. The first is that x_j comes from μ_1^f.

If $p \neq f$ and $e_f \neq i-1$, then we have

$$\begin{aligned} \partial^{e_p} \circ \mu_1^{e_f}(x) &= (-1)^{f+1+p} x_{e_p} x_j (x_i x_f; e_1, \dots, \widehat{e_p}, \dots, e_{f-1}) \\ &= -\mu_1^{e_f} \circ \partial^{e_p}(x)\,. \end{aligned}$$

If instead we have $p = f$, and $e_f, e_{f-1} \neq i-1$ then

$$\begin{aligned} \mu_1^{e_{f-1}} \circ \partial^{e_f}(x) &= (-1)^{2f} x_{e_f} x_j (x_i x_{f-1}; e_1, \dots, e_{f-2}) \\ &= -\mu_1^{e_{f-1}} \circ \mu_1^{e_f}(x)\,. \end{aligned}$$

Finally, if we have $p-1 \in \{c+1, \dots, c+r-1\}$ and $e_f > i+1$,

$$\begin{aligned} \mu_2^{e_{p-1}} \circ \mu_1^{e_f}(x) &= (-1)^{f+c+r+2} x_p x_j (x_{p-1} x_f; e_1, \dots, e_{c+1}, \dots, \widehat{e_{p-1}}, \widehat{e_p}, \dots, e_{c+r}, i, e_{c+r+1} \dots, e_{f-1}) \\ &= -\mu_1^{e_f} \circ \mu_2^{e_{p-1}}(x)\,. \end{aligned}$$

The other way that x_j can be part of the coefficient of a term of $d^2(x)$ is that it comes from $\mu_1^{e_q}$ where $e_{q+1} \neq e_q + 1$ and where $f = c+r$.

In this case, if $p < q$, we have

$$\begin{aligned} \partial^{e_p} \circ \mu_1^{e_q}(x) &= (-1)^{q+1+p} x_{e_p} x_j (x_{e_q} x_i; e_1, \dots, \widehat{e_p}, \dots, \widehat{e_q}, \dots, e_{c+1}, \dots, e_{c+r}) \\ &= -\mu_1^{e_q} \circ \partial^{e_p}(x)\,. \end{aligned}$$

The case where $p > q$ is similar and results in the same relation.

In addition, if $p < q$ and $e_p = e_{p-1} + 1$, $e_{p+1} = e_p + 1$, …, $e_q = e_{q-1} + 1$, then we have

$$\begin{aligned}\mu_2^{e_{p-1}} \circ \mu_1^{e_q}(x) &= (-1)^{2q+1} x_{e_p} x_j (x_{e_{p-1}} x_i; e_1, \ldots, \widehat{e_{p-1}}, \widehat{e_p}, \ldots, e_q, \ldots, e_{c+1}, \ldots, e_{c+r})) \\ &= -\mu_1^{e_{p-1}} \circ \partial^{e_p}(x)\,.\end{aligned}$$

Case 6. Finally we consider terms of $d^2(x)$ whose coefficients are $x_i x_j$. First note that the variable x_i only divides the coefficient of terms which come from μ_1 or terms which come from $\partial^i \circ \mu_2$. However, the coefficient of $\partial^i \circ \mu_2^{e_p}(x) \neq x_i x_j$ for any p. This together with the fact that x_j only appears as part of a coefficient via the map μ_1 means that $x_i x_j$ only appears as the coefficient of terms of μ_1^2.

Hence the only terms of $d(x)^2$ which have coefficient $x_i x_j$ appear in two cases. The first case is when $f = c + r$, $e_{q+1} \neq e_q + 1$. In this case we have

$$\begin{aligned}\mu_1^{e_{c+r}} \circ \mu_1^{e_q}(x) &= (-1)^{q+1+c+r} x_i x_j (x_{e_q} x_{i-1}; e_1, \ldots, \widehat{e_q}, \ldots, e_{c+1}, \ldots, e_{c+r-1}) \\ &= -\mu_1^{e_q} \circ \mu_1^{e_{c+r}}(x)\,.\end{aligned}$$

The other case in which we have terms with the coefficient $x_i x_j$ is when $f \neq c + r$. In this case we have

$$\begin{aligned}\mu_1^{e_f} \circ \mu_1^{e_{c+r}}(x) &= (-1)^{c+r+1+f} x_i x_j (x_{i-1} x_{e_f}; e_1, \ldots, \widehat{e_{c+r}}, \ldots, e_{f-1}) \\ &= -\mu_1^{e_{c+r}} \circ \mu_1^{e_f}(x)\,.\end{aligned}$$

□

Theorem 2.2. *The minimal free resolution of S/J_n has basis 1 in homological degree 0 and basis $(x_i x_j; e_1, \ldots, e_c, e_{c+1} \ldots, e_{c+r}, \ldots, e_f)$ in homological degree $f+1$ where $x_i x_j$ is a minimal generator of J_n and $e_1 < e_2 < \cdots < e_{c+r} < i < i+1 < e_{c+r+1} < \cdots < e_f < j$, $e_{c+1} = i - r, e_{c+2} = i - r + 1, \ldots, e_{c+r} = i - 1$, and $e_c \neq i - r - 1$. The differential of the resolution is the map d defined above.*

Proof. We prove this by induction on n. First consider the case where $n = 4$. The minimal free resolution, $\mathbf{G}$, of S/J_4 is the following

$$0 \longrightarrow S^2 \xrightarrow{d_1} S^3 \xrightarrow{d_0} S \longrightarrow 0\,,$$

where the basis of G_1 is

$$\Big\{(x_1x_3;\emptyset), (x_1x_4;\emptyset), (x_2x_4;\emptyset), \Big\}\,,$$

and the basis of G_2 is

$$\Big\{(x_1x_4;3), (x_2x_4;1)\Big\}\,.$$

The differential of $\mathbf{G}$ is given by the following two maps:

$$d_0 = \begin{pmatrix} x_1x_3 & x_1x_4 & x_2x_4 \end{pmatrix}$$

$$d_1 = \begin{pmatrix} x_4 & 0 \\ -x_3 & x_2 \\ 0 & -x_1 \end{pmatrix}.$$

It is easily checked that this is exact and hence it is the minimal free resolution of S/J_4.

Now assume that the minimal free resolution of S/J_{n-1} is as stated. Call this minimal free resolution $\mathbf{F}$. We will construct the minimal free resolution of S/J_n by using a series of mapping cones; one for each of the minimal monomial generators $x_1x_n, \ldots, x_{n-2}x_n$. First we have the following short exact sequence:

$$0 \longrightarrow S/(J_{n-1} : x_1x_n) \xrightarrow{x_1x_n} S/J_{n-1} \longrightarrow S/(J_{n-1} + (x_1x_n)) \longrightarrow 0\,.$$

The ideal $J_n + (x_1x_n)$ is the edge ideal of the complement of the n-cycle with the additional edge $\{1, n\}$. In this graph, the edge $\{1, n\}$ is a splitting edge as defined by Hà and Van Tuyl in [HVT07]. In their paper they study the effect on the edge ideal of removing a splitting edge from a graph as in the short exact sequence above. In the remainder of this proof we will have similar short exact sequences for each minimal monomial generator x_ux_n, however only this first short exact sequence and the last, (that corresponding to the final minimal generator $x_{n-2}x_n$), are examples of short exact sequences representing the removal of a splitting edge.

Note that the ideal $(J_{n-1} : x_1x_n) = (x_3, x_4, \ldots, x_{n-1})$. Then the minimal free resolution of $S/(J_{n-1} : x_1x_n)$ is the Koszul complex on the variables $\{x_3, x_4, \ldots, x_{n-1}\}$. Call this Koszul complex $\mathbf{K}^{(1)}$ and shift the multigrading so that the generator in homological degree 0 has multidegree x_1x_n. We denote the generator in homological degree 0 of $\mathbf{K}^{(1)}$ by $(x_1x_n; \emptyset)$, and the basis in homological degree $f \geq 1$ by

$$\Big\{(x_1x_n; e_1, e_2, \ldots, e_f)\Big|\ 3 \leq e_1 < \cdots < e_f \leq n-1\Big\}.$$

The differential of $\mathbf{K}^{(1)}$ is given by ∂ as we have defined it above.

Let $\mu = \mu_1 + \mu_2$, and extend μ so that $\mu((x_1x_n; \emptyset)) = -x_1x_n$. It is easy to see that $\partial^2 = 0$, so by Lemma 2.1 we have $\mu \circ \partial = -d \circ \mu$. Thus the map $(-\mu) : \mathbf{K}^{(1)} \longrightarrow \mathbf{F}$ is a map of complexes of degree 0 which lifts the map $S/(J_{n-1} : x_1x_n) \xrightarrow{x_1x_n} S/J_{n-1}$.

The mapping cone of $(-\mu) : \mathbf{K}^{(1)} \to \mathbf{F}$ gives us a minimal free resolution of $S/(J_{n-1} + (x_1x_n))$ with differential $\partial + \mu$. Call this resolution $\mathbf{F}^{(1)}$.

For each of the minimal monomial generators $x_1x_n, x_2x_n, \ldots, x_{n-2}x_n$ of J_n we have a similar short exact sequence and mapping cone. We show the step which adds the minimal monomial generator x_ux_n. Let $\mathbf{F}^{(u-1)}$ be the minimal free resolution of $S/(J_{n-1} + (x_1x_n, x_2x_n, \ldots, x_{u-1}x_n))$ obtained in the previous step. The basis of $\mathbf{F}^{(u-1)}$ in degree $f + 1$ is

$$\Big\{(x_ix_j; e_1, \ldots, e_c, \ldots, e_f)\Big| e_1 < e_2 < \cdots < e_c < i,\ i+1 < e_{c+1} < \cdots < e_f\Big\}$$

where x_ix_j is a minimal generator of the ideal $(J_{n-1} + (x_1x_n, \ldots, x_{u-1}x_n))$.

We have the short exact sequence:

$$0 \longrightarrow S/((J_{n-1} + (x_1x_n, \ldots, x_{u-1}x_n)) : x_ux_n) \xrightarrow{x_ux_n} S/(J_{n-1} + (x_1x_n \ldots, x_{u-1}x_n)) \longrightarrow S/(J_{n-1} + (x_1x_n, \ldots, x_ux_n)) \longrightarrow 0\,.$$

Note that $(J_{n-1} + (x_1x_n, \ldots, x_{u-1}x_n) : x_ux_n) = (x_1, x_2, \ldots, x_{u-1}, x_{u+2}, \ldots, x_n)$. Let $\mathbf{K}^{(u)}$ be the Koszul complex on the elements $\{x_1, x_2, \ldots, x_{u-1}, x_{u+2}, \ldots, x_n\}$. We multigrade this complex so that the basis element in homological degree 0 has multidegree x_ux_n. $\mathbf{K}^{(u)}$ has differential ∂ and basis in homological degree f given by

$$\left\{(x_ux_n; e_1, \ldots, e_c \ldots, e_f) \middle| e_1 < e_2 < \cdots < e_c < u, u+1 < e_{c+1} < \cdots < e_f < n\right\}.$$

As before, we define $\mu(x_ux_n; \emptyset) = -x_ux_n$. Then the map $(-\mu) : \mathbf{K}^{(u)} \longrightarrow \mathbf{F}^{(u-1)}$ is a map of complexes of degree 0 which lifts the map $S/(J_{n-1} + (x_1x_n, \ldots, x_{u-1}x_n) : x_ux_n) \xrightarrow{x_ux_n} S/(J_{n-1} + (x_1x_n, \ldots, x_{u-1}x_n))$. Let $\mathbf{F}^{(u)}$ be the mapping cone complex of this map of complexes. $\mathbf{F}^{(u)}$ is a free resolution of $S/(J_{n-1} + (x_1x_n, \ldots, x_ux_n))$. This resolution is minimal since the basis elements in homological degree $f > 0$ all have multidegree a monomial of degree $f + 1$.

□

Next we construct a regular cellular structure which supports the minimal free resolution of S/J_n which we have just constructed.

Theorem 2.3. *There exists a regular cell complex supporting the minimal free resolution of the ideal S/J_n for all $n \geq 4$.*

Proof. We proceed by induction on n. A regular cell complex supporting the minimal free resolution of $S/J_4 = S/(x_1x_3, x_1x_4, x_2x_4)$ is shown in Figure 1.

We use the same notation as in the proof of Theorem 2.2: $\mathbf{F}$ is the minimal free resolution of S/J_{n-1} with basis and differential as in Theorem 2.2, $\mathbf{F}^{(u)}$ the minimal free resolution of $S/(J_{n-1} + (x_1x_n, \ldots, x_ux_n))$, and $\mathbf{K}^{(u)}$ the Koszul complex on the variables $\{x_1, \ldots, x_{u-1}, x_{u+2}, \ldots, x_{n-1}\}$ shifted so that the generator in homological degree 0 has multidegree x_ux_n.

Let X_{n-1} be a regular cellular complex supporting S/J_{n-1}. We will construct a regular cellular complex supporting the minimal free resolution of S/J_n by constructing a regular cellular complex $X_{n-1}^{(u)}$ supporting the resolution $\mathbf{F}^{(u)}$ for each $1 \leq u \leq n-2$ in turn.

Recall from the proof of Theorem 2.2 that $\mathbf{F}^{(1)}$ is the mapping cone of the map $(-\mu)$: $\mathbf{K}^{(1)} \longrightarrow \mathbf{F}$ where $\mathbf{K}^{(1)}$ is the Koszul complex on the variables $\{x_3, \ldots, x_{n-1}\}$. The Koszul complex $\mathbf{K}^{(1)}$ is supported on an $(n-4)$-dimensional simplex with vertices labeled by the basis elements

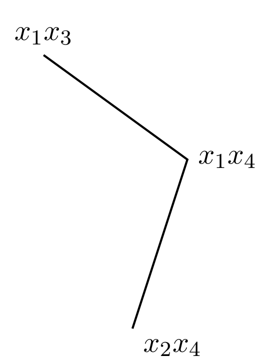

Figure 1: A regular cell complex supporting the minimal free resolution of S/J_4.

$(x_1x_n;x_3)\ldots,(x_1x_n;x_{n-1}$. Since the mapping cone construction shifts the basis elements of $\mathbf{K}^{(1)}$ up a homological degree, these vertices become the new one-dimensional cells. The 1-cell $(x_1x_n;x_i)$ has endpoints $(x_1x_n;\emptyset)$ and $(x_1x_i;\emptyset)$. Thus adding $\mathbf{K}^{(1)}$ to $\mathbf{F}$ corresponds to adding a cone over the point $(x_1x_n;\emptyset)$ to X_{n-1}. This cone is attached to X_{n-1} at the cell $(x_1x_{n-1};3,\ldots n-2)$ since

$$\mu((x_1x_n;3,\ldots,n-1)) = (-1)^{n-2}x_n(x_1x_{n-1};3,\ldots,n-2)\,.$$

Let $X^{(1)}_{n-1}$ be X_{n-1} together with this cone over the point $(x_1x_n;\emptyset)$ with base the cell $(x_1x_{n-1};3,\ldots,n-2)$. Since X_{n-1} was regular and since the base of the cone we have just added is a single $(n-4)$-dimensional cell, the complex X^1_{n-1} is a regular cell complex which supports the resolution $\mathbf{F}^{(1)}$.

Now suppose that we have constructed a regular cell complex, $X^{(u-1)}_{n-1}$ supporting the resolution $\mathbf{F}^{(u-1)}$. We wish to construct a regular cellular complex $X^{(u)}_{n-1}$ supporting $\mathbf{F}^{(u)}$. We obtain $\mathbf{F}^{(u)}$ from the mapping cone of the map $(-\mu):\mathbf{K}^{(u)}\longrightarrow \mathbf{F}^{(u-1)}$.

The Kozsul complex $\mathbf{K}^{(u)}$ is supported on an $(n-4)$-dimensional simplex with vertices labeled by the basis elements $\{(x_ux_n;j)|j\in\{1,\ldots,u-1,u+2,\ldots,n-1\}\}$. Again, the mapping cone construction shifts the basis elements of $\mathbf{K}^{(u)}$ up a homological degree so that these vertices become the new 1-cells. The 1-cell $(x_ux_n;j)$ has endpoints $(x_ux_n;\emptyset)$ and $(x_jx_u;\emptyset)$ for $j\neq u-1$ and for $j=u-1$ the cell $(x_ux_n;j)$ has endpoints $(x_ux_n\emptyset)$ and $(x_jx_n;\emptyset)$. Adding $\mathbf{K}^{(u)}$ to $\mathbf{F}^{(u-1)}$ thus corresponds to adding a cone over the point $(x_ux_n;\emptyset)$. The base of this cone is the collection of cells in $X^{(u-1)}_{n-1}$ which are labelled by the basis elements of $\mathbf{F}^{(u-1)}$ which make up $\mu(x_ux_n;1,2,\ldots,u-1,u+2,\ldots,n-1)$. In other words, the base of the cone is the collection of cells

$$\begin{gathered}(x_ux_{n-1};1,2,\ldots,u-1,u+2,\ldots,n-2),\\ (x_1x_n;3,\ldots,u,u+2,\ldots,n-1),\\ (x_2x_n;1,4,\ldots,u-1,u,u+2,\ldots,n-1),\\ \vdots\\ (x_{u-2}x_n;1,\ldots,u-3,u,u+2,\ldots,n-1),\\ (x_{u-1}x_n;1,\ldots,u-2,u+2,\ldots,n-1).\end{gathered}$$

Let $X^{(u)}_{n-1}$ be the regular cell complex $X^{(u-1)}_{n-1}$ together with this cone. In order to show that $X^{(u)}_{n-1}$ is regular we need only show that the union of the cells labelled by

$$\begin{gathered}(x_ux_{n-1};1,2,\ldots,u-1,u+2,\ldots,n-2),\\ (x_1x_n;3,\ldots,u,u+2,\ldots,n-1),\\ (x_2x_n;1,4,\ldots,u-1,u,u+2,\ldots,n-1),\\ \vdots\\ (x_{u-2}x_n;1,\ldots,u-3,u,u+2,\ldots,n-1),\\ (x_{u-1}x_n;1,\ldots,u-2,u+2,\ldots,n-1)\end{gathered}$$

in the cell complex $X^{(u-1)}_{n-1}$ is homeomorphic to an $(n-4)$-dimensional ball.

First consider just the first two elements in this list. The intersection of these two elements is

$$\begin{aligned}\mu^1_2((x_ux_{n-1};1,2,\ldots,u-1,u+2,\ldots,n-2)) &= (x_1x_{n-1};3,4,\ldots,u-1,u,u+2,\ldots,n-2)\\ &= \partial^{n-1}((x_1x_n;3,\ldots,u,u+2,\ldots,n-1))\end{aligned}$$

if $u > 2$, and

$$\begin{array}{rcl}\mu_1^1((x_2x_{n-1};1,4,\ldots,n-2)) & = & (x_1x_{n-1};4,\ldots,n-2) \\ & = & \partial^{n-1}((x_1x_n;4\ldots,n-1))\end{array}$$

if $u = 2$. (We have already considered the case where $u = 1$). In either case the intersection consists of a single cell of dimension $n-5$. This is homeomorphic to an $(n-5)$-ball and thus the union of the two elements $(x_ux_{n-1};1,2,\ldots,u-1,u+2,\ldots,n-2)$ and $(x_1x_n;3,\ldots,u,u+2,\ldots,n-1)$ is homeomorphic to an $(n-4)$-ball.

Now suppose that we know that the union of the first p elements in the list are homeomorphic to an $(n-4)$-ball. Explicitly, we assume that the union of the cells

$$\begin{array}{c}(x_ux_{n-1};1,2,\ldots,u-1,u+2,\ldots,n-2), \\ (x_1x_n;3,\ldots,u,u+2,\ldots,n-1), \\ (x_2x_n;1,4,\ldots,u-1,u,u+2,\ldots,n-1), \\ \vdots \\ (x_{p-1}x_n;1,\ldots,p-2,p+1,\ldots,u-1,u,u+2,\ldots,n-1)\end{array}$$

is homeomorphic to an $(n-4)$-dimensional ball.

The intersection of the cell $(x_px_n;1,\ldots,p-1,p+2,\ldots,u-1,u,u+2,\ldots,n-1)$ with the union of cells listed above is the following union of cells:

$$\begin{array}{c}(x_px_{n-1};1,\ldots,p-1,p+2,\ldots,u,u+2,\ldots,n-2), \\ (x_1x_n;3,\ldots,p,p+2,\ldots,u,u+2,\ldots,n-1), \\ (x_2x_n;1,4,\ldots,p,p+2,\ldots,u,u+2,\ldots,n-1), \\ \vdots \\ (x_{p-1}x_n;1,\ldots,p-2,p+2,\ldots,u,u+2,\ldots,n-1)\,.\end{array}$$

These cells are the collection of cells which come from $\mu(\partial^{u+1}(x_px_n;1,\ldots,p-1,p+2,\ldots,n-1))$. Since $(x_px_n;1,\ldots,p-1,p+2,\ldots,n-1)$ is a regular cell which is a cone over the point $(x_px_n;\emptyset)$, the face $\partial^{u+1}((x_px_n;1,\ldots,p-1,p+2,\ldots,n-1))$ is also a regular cell which is a cone over the point $(x_px_n;\emptyset$. Therefore the base cells of this cone (i.e. the cells of $\mu(\partial^{u+1}(x_px_n;1,\ldots,p-1,p+2,\ldots,n-1)))$ must be homeomorphic to an $(n-5)$-ball and thus the union of the set of cells

$$\begin{array}{c}(x_ux_{n-1};1,2,\ldots,u-1,u+2,\ldots,n-2), \\ (x_1x_n;3,\ldots,u,u+2,\ldots,n-1), \\ (x_2x_n;1,4,\ldots,u-1,u,u+2,\ldots,n-1), \\ \vdots \\ (x_{p-1}x_n;1,\ldots,p-2,p+1,\ldots,u-1,u,u+2,\ldots,n-1)\end{array}$$

and the cell $(x_px_n;1,\ldots,p-1,p+2,\ldots,u-1,u,u+2,\ldots,n-1)$ is homeomorphic to an $(n-4)$-ball.

□

Example 2.4. In Figure 2 we show the steps in constructing the regular cell structure supporting the minimal free resolution of S/J_5 from that supporting the minimal free resolution of S/J_4 (shown in Figure 1). Part (a) of Figure 2 shows the regular cell structure supporting the minimal free resolution of S/J_4. The first step in the construction adds a cone over the point $(x_1x_5;\emptyset)$ with base the cell $(x_1x_4;3)$. This step is shown in Figure 2 (b).

The next step of the construction adds a cone over the point $(x_2x_5;\emptyset)$ with base the union of the cells $(x_2x_4;1)$ and $(x_1x_5;4)$. This step is shown in Figure 2 (c).

The final step in the construction is shown in Figure 2 (d). It adds a cone over the point $(x_3x_5;\emptyset)$ with base the union of the cells $(x_2x_5;1)$ and $(x_1x_5;3)$.

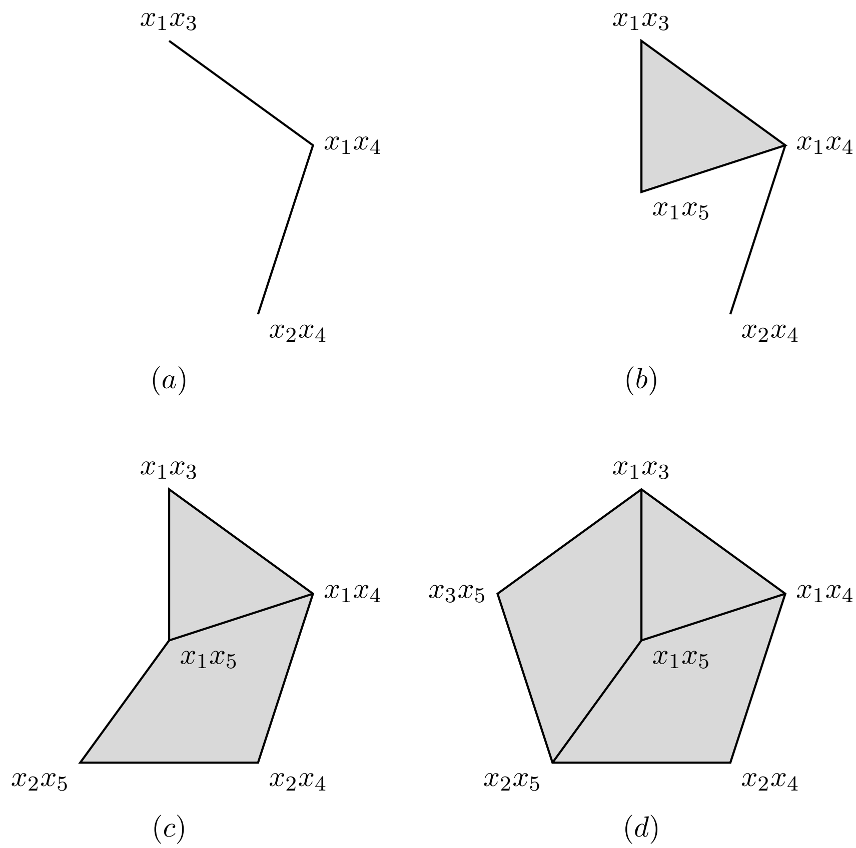

Figure 2: The construction of a regular cell complex supporting the minimal free resolution of S/J_5.

Definition 2.5. We say a CW-complex, X, is *pure of dimension* d if every cell of X is contained in the boundary of a cell of dimension d.

Proposition 2.6. *The regular cell complex* X_n *constructed in Theorem 2.3 which supports the minimal free resolution of* S/J_n *is pure of dimension* $n-3$.

Proof. We prove this by induction on n. It is clear from Figure 1 that the regular cell complex supporting the minimal free resolution of S/J_4 is pure of dimension 1. Now let X_n be the regular cell complex supporting the minimal free resolution of S/J_n and suppose that the regular cell complex X_{n-1} supporting S/J_{n-1} is pure of dimension $n-4$. By the way we constructed X_n from X_{n-1} every cell of X_n which was not in X_{n-1} is contained in the boundary of an $(n-3)$-dimensional cell. Therefore, to finish the proof we need to show that every cell of X_{n-1} is contained in an

$(n-3)$-dimensional cell in X_n. Since X_{n-1} is pure of dimension $n-4$, we only need to consider the $(n-4)$-dimensional cells of X_{n-1}.

Every $(n-4)$-dimensional cell of X_{n-1} has the form $(x_ix_{n-1}; 1, 2, \ldots, i-1, i+2, \ldots, n-2)$ for some $1 \leq i \leq n-3$. Then

$$\mu_1^{n-1}((x_ix_n; 1, 2, \ldots, i-1, i+2, \ldots, n-1)) = (-1)^{n-2}x_n(x_ix_{n-1}; 1, 2, \ldots, i-1, i+2, \ldots, n-2)\,,$$

so $(x_ix_{n-1}; 1, 2, \ldots, i-1, i+2, \ldots, n-2)$ is part of the boundary of the $(n-3)$-dimensional cell $(x_ix_n; 1, 2, \ldots, i-1, i+2, \ldots, n-1)$ in X_n. Hence X_n is pure of dimension $n-3$.

□

3 The resolution of S/I_n

In this section we construct a regular cell complex which supports the minimal free resolution of S/I_n. We do this by taking the cells from the regular cell complex supporting S/J_n which we have already constructed which do not contain the point x_1x_n and then adding an additional cell. We then show that the resulting complex satisfies the necessary acyclicity conditions so that it supports the minimal free resolution of S/I_n.

Before we construct the regular cell complex supporting the minimal free resolution of S/I_n, we need to know more about the structure of the regular cell complex we constructed to support the minimal free resolution of S/J_n. To this end, we need the following lemma and proposition.

Lemma 3.1. *The cells of X_n which contain as part of their boundary the point $(x_1x_n; \emptyset)$ are exactly those cells which are labeled by symbols of the form $(x_ix_n; 1, 2, \ldots, i-1, e_i, e_{i+1}, \ldots, e_f)$ where $i+2 \leq e_i < e_{i+1} < \cdots < e_f < n$.*

Proof. One direction of this claim is easy. Any cell of the form $(x_ix_n; 1, 2, \ldots, i-1, e_i, e_{i+1}, \ldots, e_f)$ contains in its boundary a cell of the form $(x_1x_n; t_1, \ldots, t_d)$. To see this, note that if $i = 1$ then the original cell is already of this form. If not, then applying μ_1^1 (if $i = 2$), or μ_2^1 (if $i > 2$) yields a cell of the desired form. Then repeated applications of ∂ to $(x_1x_n; t_1, \ldots, t_d)$ will eventually yield $(x_1x_n; \emptyset)$.

We prove the opposite direction by induction on the dimension of the cell. Clearly the only 1-dimensional cells which contain $(x_1x_n; \emptyset)$ in their boundary are cells of the form $(x_1x_n; j)$ for some $3 \leq j \leq n-1$ and the cell $(x_2x_n; 1)$.

Now suppose that the claim holds for cells of dimension $f-1$. Let $x = (x_ix_j; e_1, \ldots, e_f)$ be a cell of dimension f which contains $(x_1x_n; \emptyset)$ as part of its boundary. In order for $(x_1x_n; \emptyset)$ to be part of the boundary of x it must be part of the boundary of one of the cells which make up $d(x)$. Let y be a cell which contains $(x_1x_n; \emptyset)$ and appears as a term of $d(x)$. Since y is a cell of dimension $f-1$, by the induction hypothesis it must be of the form $y = (x_ux_n; 1, 2, \ldots, u-1, t_u, t_{u+1}, \ldots, t_{f-1})$ with $u+1 < t_u < t_{u+1} < \cdots < t_{f-1} < n$.

In order for y to be a term of $d(x)$, either y is a term of $\partial(x)$ or y is a term of $\mu(x)$. If y is a term of $\partial(x)$, x must have the form $(x_ux_n; 1, 2, \ldots, u-1, e_u, \ldots, e_f)$ with $\{t_u, t_{u+1}, \ldots, t_{f-1}\} \subset \{e_u, e_{u+1}, \ldots, e_f\}$.

Since x_n divides the multidegree of y, if y is a term of $\mu(x)$ then $x = (x_ix_n; e_1, \ldots, e_f)$. In order for $\mu_2(x)$ to be non-zero, x must have the form $(x_ix_j; 1, 2, \ldots, i-1, e_i, e_{i+1}, \ldots, e_f)$. So if y is a term of $\mu_2(x)$, $x = (x_ix_n; 1, 2, \ldots, i-1, e_i, e_{i+1}, \ldots, e_f)$ with $i > u+1$. Finally, since $b(x_ux_ix_n) = x_ux_n$ if and only if $i = u+1$, if y is a term of $\mu_1(x)$ then we must have $x = (x_{u+1}x_n; 1, 2, \ldots, v, e_{u+1}, \ldots, e_f)$. Thus, in order for $(x_1x_n; \emptyset)$ to be contained in the boundary of x, x must be of the form $x = (x_ix_n; 1, \ldots, i-1, e_i, e_{i+1}, \ldots, e_f)$.

□

Proposition 3.2. *Let X_n be the regular cell complex supporting the minimal free resolution of S/J_n which was constructed in Theorem 2.3. Then the boundary of the union of the $(n-3)$-dimensional cells of X_n is homeomorphic to a sphere of dimension $n-4$.*

Proof. The dimension $n-3$ cells of X_n correspond to the following basis elements of the minimal free resolution of S/J_n:

$$\begin{gathered}(x_1x_n; 3, 4, \ldots, n-1),\\ (x_2x_n; 1, 4, \ldots, n-1),\\ \vdots\\ (x_px_n; 1, 2, \ldots, p-1, p+2, \ldots, n-1),\\ \vdots\\ (x_{n-2}x_n; 1, 2, \ldots, n-3)\,.\end{gathered}$$

By Lemma 3.1, all of these cells contain the point $(x_1x_n; \emptyset)$. Any two of these $(n-3)$-dimensional cells intersect in exactly one $(n-4)$-dimensional cell which also contains the point $(x_1x_n; \emptyset)$. More explicitly, the intersection of the cells $(x_px_n; 1, 2, \ldots, p-1, p+2, \ldots, n-1)$ and $(x_qx_n; 1, 2, \ldots, q-1, q+2, \ldots, n-1)$ where $p < q$ is exactly the $(n-4)$-dimensional cell $(x_px_n; 1, 2, \ldots, p-1, p+2, \ldots, \widehat{q+1}, \ldots, n-1)$. Conversely, every $(n-4)$-dimensional cell which contains the point $(x_1x_n; \emptyset)$ is of the form $(x_px_n; 1, 2, \ldots, p-1, p+2, \ldots, \widehat{q+1}, \ldots, n-1)$ and thus is contained in boundary of exactly two $(n-3)$-dimensional cells. On the other hand, an $(n-4)$-dimensional cell which does not contain $(x_1x_n; \emptyset)$ can have two forms. It is either of the form $(x_px_{n-1}; 1, 2, \ldots, p-1, p+2, \ldots, n-2)$ or of the form $(x_px_n; 1, \ldots, \widehat{q}, \ldots, p-1, p+2, \ldots, n-1)$. In either of these cases the $(n-4)$-dimensional cells is contained in exactly one $(n-3)$-cell, $(x_px_n; 1, 2, \ldots, p-1, p+2, \ldots, n-1)$. This structure together with the fact that X_n is contractible means that X_n is homeomorphic to an $(n-3)$-dimensional ball. Therefore the boundary of X_n, by which we mean the $(n-4)$-cells which are contained in only one $(n-3)$-dimensional cell is homeomorphic to an $(n-4)$-sphere.

□

Now we are ready to construct a CW-complex which supports the minimal free resolutionof S/I_n.

Construction 3.3. Define a CW-complex Y_n as follows:

The dimension 0 cells of Y_n are the dimension 0 cells of X_n minus the 0-cell $(x_1x_n; \emptyset)$. The dimension f cells of Y_n are the dimension f cells of X_n which do not contain the point $(x_1x_n; \emptyset)$ in their boundary for $1 \leq f \leq n-4$. There is one dimension $n-3$ cell of Y_n whose boundary is the union of all the dimension $n-4$ cells of Y_n.

Before proving that Y_n supports the minimal free resolution of S/I_n we will need the following definition.

Definition 3.4. Let X be a CW-complex whose 0-cells are labeled by monomials and whose higher dimensional cells are labeled by the lcm of the monomials labeling the 0-cells contained in the boundary of the given cell. For a monomial m define $X_{\leq m}$ to be the subcomplex of X consisting of all cells labeled by monomials which divide m.

Theorem 3.5. *The minimal free resolution of S/I_n is supported on the regular cellular complex Y_n.*

Proof. We need only show that for every monomial m in the lcm lattice of I_n, the subcomplex $(Y_n)_{\leq m}$ is acyclic. Let m be an element of the lcm lattice of I_n which is not the product of all the variables $x_1, \ldots, x_n$. If x_1x_n does not divide m, then $(Y_n)_{\leq m} = (X_n)_{\leq m}$. Since the CW-complex X_n supports the minimal free resolution of S/J_n, we know that $(X_n)\leq m$ is acyclic.

Now suppose that x_1x_n does divide m. Let $m = x_1x_2 \cdot \cdots \cdot x_ix_{e_{i+1}} \cdot \cdots \cdot x_{e_f}$. Then by Lemma 3.1 the only cell of X_n which has multidegree m and contains the point x_1x_n in its boundary is the cell $x = (x_ix_n; 1, 2, \ldots, i-1, e_{i+1}, \ldots, e_f)$. Since we got Y_n from X_n by taking the cells which did not contain the point $(x_1x_n; \emptyset)$, $(X_n)_{\leq m} = (Y_n)_{\leq m} \cup x$ where x is attached to $(Y_n)_{\leq m}$ along the cells of the boundary of x which do not contain the point $(x_1x_n; \emptyset)$. Since $(X_n)_{\leq m}$ is contractible, if we knew that the intersection of the cell x with $(Y_n)_{\leq}$ was also contractible, then $(Y_n)_{\leq m}$ would have to be contractible as well.

To see that the union of the cells of the boundary of x containing the point $(x_1x_n; \emptyset)$ is contractible, suppose that y and z are two cells contained in the boundary of x such that $(x_1x_n; \emptyset)$ is contained in the boundary of both y and z. Further suppose that $w = (x_ux_v; p_1, \ldots, p_c)$ is a cell contained in the intersection of y and z. Since both y and z contain $(x_1x_n; \emptyset)$, they have the form $y = (x_{j_1}x_n; 1, 2, \ldots, i_1-1, t_1, \ldots, t_{f_1})$ and $z = (x_{j_2}x_n; 1, 2, \ldots, i_2-1, s_1, \ldots, s_{f_2})$. Let the lcm of the multidegree of y and the multidegree of z be $x_1x_2 \cdot \cdots \cdot x_jx_{e_1} \cdot \cdots \cdot x_{e_f}$ where $j = \min\{j_1, j_2\}$. Then it is not hard to check that w is contained in the cell $(x_jx_n; 1, 2, \ldots, j-1, e_1, \ldots, e_f)$ which is also contained in the intersection of y and z. Since all of the cells in the boundary of x which contain the point $(x_1x_n; \emptyset)$ intersect in cells which also contain $(x_1x_n; \emptyset)$, the union of cells in the boundary of x which contain $(x_1x_n; \emptyset)$ is contractible. Since X_n is a regular CW-complex, the boundary of x is homeomorphic to a sphere, therefore the union of the cells of the boundary of x which do not contain $(x_1x_n; \emptyset)$ is also contractible.

Finally, we must check that $(Y_n)_{\leq x_1 \cdot \cdots \cdot x_n}$ is acyclic as well. By construction of Y_n, we know $(Y_n)_{\leq x_1 \cdot \cdots \cdot x_n} = Y_n$, which consists of a single $(n-3)$-dimensional cell whose boundary is homeomorphic to a sphere. Therefore, Y_n is acyclic, and we are done.

□

We end with two examples of regular cell complexes which support the minimal free resolution of S/I_4 and S/I_5.

Example 3.6. The regular cell complex which we constructed in Theorem 2.3 which supports the minimal free resolution of S/J_4 is shown in Figure 3 (a). A regular cell complex which supports the minimal free resolution of S/I_4 is obtained from this cell complex by removing the cells which contain the 0-cell $(x_1x_4; \emptyset)$ (for simplicity, in Figure 3 this cell is labeled by its multidegree x_1x_4), and adding a 1-cell whose boundary made up of the cells $(x_1x_3; \emptyset)$ and $(x_2x_4; \emptyset)$. This is shown in Figure 3 (b).

Example 3.7. The regular cell complex which we constructed in Theorem 2.3 which supports the minimal free resolution of S/J_5 is shown in Figure 4 (a). A regular cell complex which supports the minimal free resolution of S/I_5 is obtained from this cell complex by removing the cells which contain the 0-cell $(x_1x_5; \emptyset)$, and adding a 1-cell whose boundary made up of the cells $(x_1x_4; 3)$, $(x_2x_4; 1)$, $(x_2x_5; 4)$, $(x_3x_5; 2)$, and $(x_3x_5; 1)$. This complex is shown in Figure 4 (b).

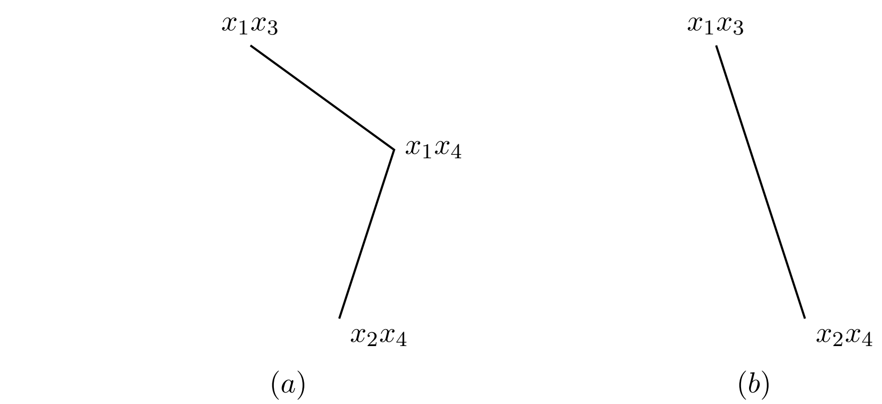

Figure 3: Regular cell complexes which support the minimal free resolutions of (a) S/J_4 and (b) S/I_4.

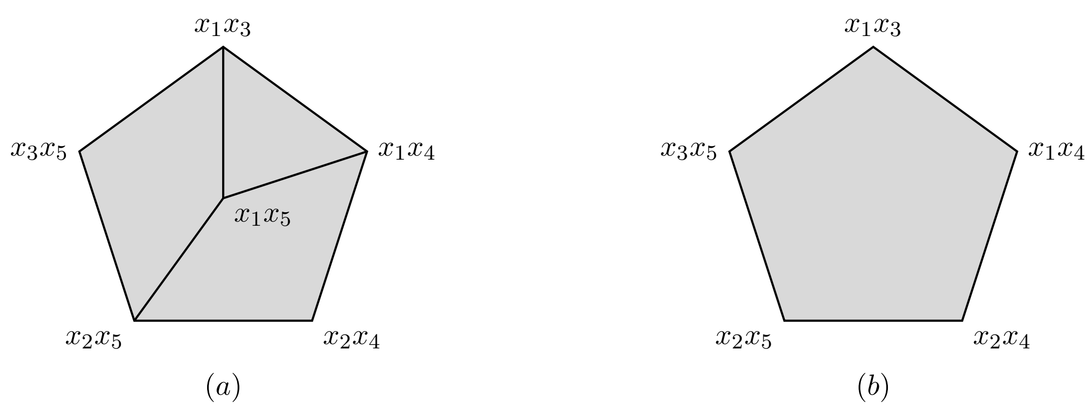

Figure 4: Regular cell complexes which support the minimal free resolutions of (a) S/J_5 and (b) S/I_5.

Remark 3.8. Let M be a monomial ideal in S. The minimal free resolution $\mathbf{F}_M$ of S/M can have more than one cellular structure. A cellular structure uses a fixed basis, and different choices of basis in $\mathbf{F}_M$ can yield different cellular structures.

The ideal J_n is an example of a specialization of a Ferrers ideal. Corso and Nagel showed in [CN08] that such an ideal is supported on a regular cell complex. However for $n \geq 5$ the regular cell complex they constructed is different than that constructed in this paper. For example, Figure 5 (a) shows the regular cell complex which supports the minimal free resolution of S/J_5 constructed in [CN08], and (b) shows that constructed in this paper.

The goal of this paper is to construct a cellular resolution of S/I_n. The cellular structure on the minimal free resolution of S/J_n is just used as a tool. The cellular structure in [CN08] cannot be used as a tool in the proof of Theorem 3.5 in the same way as we use our cellular structure. Consider how the proof of Theorem 3.5 works in the example in Figure 5. We use the cellular complex in Figure 5 (b) by removing all the cells containing the vertex x_1x_5 and then gluing a new two-dimensional cell to the remaining pentagon (the pentagon is the boundary of the new cell).

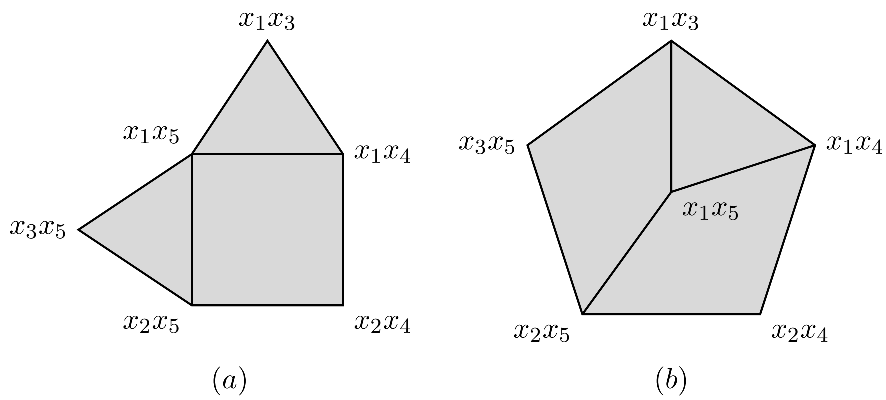

Figure 5:

If we remove the cells containing the vertex x_1x_5 from the cellular complex in Figure 5 (a), then we get the four edges $\{\{x_1x_3, x_1x_4\}, \{x_1x_4, x_2x_4\}, \{x_2x_4, x_2x_5\}, \{x_2x_5, x_3x_5\}\}$ which don't form a cycle, so we cannot glue a new two-dimensional cell to them.

It should also be noted that for small numbers of n the resolution constructed here is the same as that constructed by Horwitz in [Hor07], however for $n \geq 9$ Horwitz's resolution cannot be applied to the ideals J_n (see Example 3.18 in [Hor07]).

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