Math 2940: Prelim 1

July 12, 2016

Name:

NetID:

Instructions: This exam is 75 minutes long. It has 11 pages including the cover and 5 questions worth a total of 100 points. No written or electronic aids are allowed.

Please fully explain all your answers except when explicitly instructed otherwise. If you need more space to answer a question, use the back of the page or the blank sheet at the end of the exam. Label your work clearly if you use extra space.

Academic integrity is expected of all Cornell University students at all times, whether in the presence or absence of members of the faculty. Understanding this, I declare I shall not give, use, or receive unauthorized aid in this examination.

Please sign below to indicate that you have read and agree to these instructions.

Signature:

1:	
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Total score:	

1. (28 points) Let

$$A = \begin{bmatrix} -2 & 4 & -2 & 1 \\ 3 & -6 & 1 & 1 \\ 1 & -2 & 0 & 1 \end{bmatrix}.$$

Denote the columns of A in order by $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4$.

(a) (10 points) Find all solutions to the equation $A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Write your answer in parametric vector form.

(b) (5 points) Use part (a) to find a linear dependence relation among the vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4$.

(c) (5 points) Do the vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4$ span all of $\mathbf{R}^3$? Explain briefly why or why not.

(d) (2 points) Suppose that $b\mathbf{a}_2 + c\mathbf{a}_3 + d\mathbf{a}_4 = \mathbf{0}$ for some $b, c, d \in \mathbf{R}$. Find a vector $\mathbf{x} \in \mathbf{R}^4$ whose four entries can be expressed in terms of b, c, d such that $A\mathbf{x} = \mathbf{0}$.

(e) (6 points) Use parts (a) and (d) to show that $\mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4$ are linearly independent.

2. (20 points) Let

$$A = \begin{bmatrix} 3 & -2 & -3 \\ 1 & -1 & -2 \\ 0 & -2 & -4 \end{bmatrix}.$$

(a) (10 points) Compute A^{-1}.

(b) (5 points) Use part (a) to find all solutions to the equation $A\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}$.

(c) (5 points) Compute the determinant of A by examining the row operations that you used in part (a).

3. (18 points) Let $S : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation that rotates each vector clockwise by $90°$, and let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation that reflects each vector across the line $x_2 = -x_1$.

(a) (6 points) If $S(\mathbf{x}) = A\mathbf{x}$ and $T(\mathbf{x}) = B\mathbf{x}$, find the matrices A and B.

(b) (8 points) Compute $S(T(\mathbf{e}_1)), S(T(\mathbf{e}_2)), T(S(\mathbf{e}_1)), T(S(\mathbf{e}_2))$. Use these to find the matrices AB and BA. You may find it helpful to draw graphs.

(c) (4 points) Use geometric reasoning to explain why $A^4 = I$ and $B^2 = I$.

4. (18 points; 3 each) Given the factorization $A = LU$ below,

$$\begin{bmatrix} 2 & 1 & 2 \\ 4 & 1 & 1 \\ -4 & -3 & -7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 2 \\ 0 & -1 & -3 \\ 0 & 0 & 0 \end{bmatrix},$$

provide short answers to the following questions. No explanations are necessary.

(a) Is A invertible?

(b) Give a solution to $A\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}$. (Hint: Don't use the factorization.)

(c) How many solutions are there to $A\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}$?

(d) Is $\text{rref}(A) = \text{rref}(L)$?

(e) Is $\text{rref}(A) = \text{rref}(U)$?

(f) Can U be written as a product of elementary matrices?

5. (16 points) Let A be an $m \times n$ matrix and let B be an $n \times k$ matrix, so that AB is an $m \times k$ matrix. Label the columns of B in order by $\mathbf{b}_1, \ldots, \mathbf{b}_k$ and the columns of AB in order by $\mathbf{v}_1, \ldots, \mathbf{v}_k$.

(a) (8 points) Suppose that for a particular choice of coefficients $c_1, \ldots, c_k$,

$$c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \cdots + c_k\mathbf{b}_k = \mathbf{0} \in \mathbf{R}^n.$$

Show that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0} \in \mathbf{R}^m.$$

(b) (8 points) If the linear transformation $T(\mathbf{x}) = A\mathbf{x}$ is one-to-one, show that the converse of part (a) holds: If

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0},$$

then

$$c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \cdots + c_k\mathbf{b}_k = \mathbf{0}.$$

(This page for scratch work.)