Math 4220: Proof of Cauchy's Theorem

This supplement explains the "small squares" proof of Cauchy's Theorem in slightly more detail than was covered in class. The argument and the images are due to Kevin Houston.[1]

Theorem. *Let Γ be a simple closed contour in a domain D. Assume that the region enclosed by Γ is contained entirely within D. If f is an analytic function on D,*

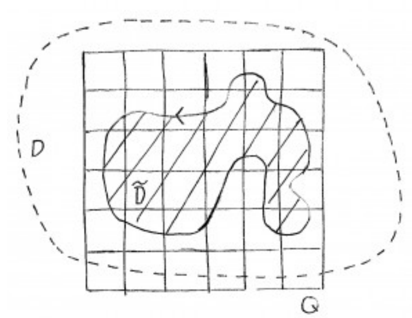
$$\int_\Gamma f(z)dz = 0.$$

Proof. We will actually show that for every $\varepsilon > 0$,

$$\left|\int_\Gamma f(z)dz\right| < \varepsilon.$$

That means the integral must be zero.

Let $\varepsilon > 0$ be fixed. We cover the region enclosed by Γ (denoted $\widetilde{D}$ in the image) by a big square Q, which we then partition into small squares Q_j having side length s.



In each square Q_j we choose a base point z_j and write

$$f(z) = f(z_j) + f'(z_j)(z - z_j) + E_j(z)$$

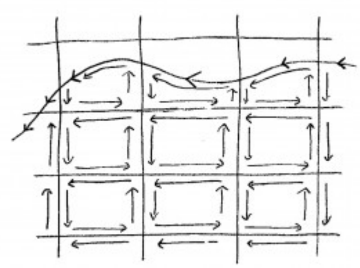
[1]Source: http://www.kevinhouston.net/blog/2013/03/what-is-the-best-proof-of-cauchys-integral-theorem/

for $z \in Q_j$. The error $E_j(z)$ is small in relation to the distance $|z - z_j|$. In fact, by making the side length s small enough, we can ensure that

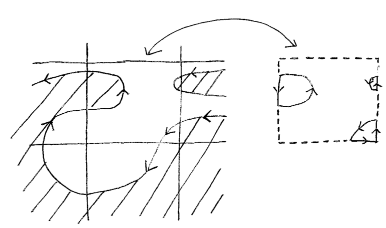
$$\left| \frac{E_j(z)}{z - z_j} \right| < \varepsilon$$

for all $z \in Q_j$. Actually there is a subtle point here that was glossed over in class. It is easy to show that for any *particular* square Q_j, by making s small enough we can force the inequality to be true. It is also true, but harder to prove, that we can pick a single value of s that works simultaneously for *all* the squares Q_j. The proof uses the notion of "compactness," for those familiar with the concept.

Continuing on, we express $\int_\Gamma f(z)dz$ as the sum of integrals over small squarish contours Γ_j with edges that cancel, as in the figure.



If Q_j is an interior square, Γ_j goes counterclockwise around the square, so it has length $\ell(\Gamma_j) = 4s$. If the curve Γ passes through Q_j, the contour Γ_j may be rather complicated, as shown below. (In this picture, Γ_j is actually the combination of three small contours, and not a single contour at all!)



Whether Γ_j comes in one piece or several pieces, its total length is at most $4s + \ell(\Gamma \cap Q_j)$, where $\ell(\Gamma \cap Q_j)$ is the length of the original curve Γ that is contained in the square Q_j.

Using the linearization inside each square,

$$\begin{aligned}\int_{\Gamma_j} f(z)dz &= \int_{\Gamma_j} [f(z_j) + f'(z_j)(z - z_j) + E_j(z)]dz \\ &= \int_{\Gamma_j} [f(z_j) + f'(z_j)(z - z_j)]dz + \int_{\Gamma_j} E_j(z)dz.\end{aligned}$$

The first integral is zero because the function $f(z_j) + f'(z_j)(z - z_j)$ has antiderivative $f(z_j)z + \frac{1}{2}f'(z_j)(z - z_j)^2$, and Γ_j is either a closed contour or a combination of closed contours (as in the figure above). Also,

$$\left|\int_{\Gamma_j} E_j(z)dz\right| \leq \max\{|E_j(z)| : z \in Q_j\} \cdot \ell(\Gamma_j).$$

The farthest away that two points in Q_j can be is $s\sqrt{2}$. Therefore,

$$\left|\frac{E_j(z)}{z - z_j}\right| < \varepsilon \Longrightarrow |E_j(z)| < \varepsilon|z - z_j| \leq \varepsilon s\sqrt{2}.$$

We get

$$\left|\int_{\Gamma_j} E_j(z)dz\right| \leq \varepsilon s\sqrt{2} \cdot [4s + \ell(\Gamma \cap Q_j)] = \varepsilon(4\sqrt{2})s^2 + \varepsilon s\sqrt{2} \cdot \ell(\Gamma \cap Q_j).$$

The total integral $\int_\Gamma f(z)dz$ is the sum over all the squares of $\int_{\Gamma_j} f(z)dz = \int_{\Gamma_j} E_j(z)dz$. Therefore,

$$\left|\int_\Gamma f(z)dz\right| \leq \sum_j \left|\int_{\Gamma_j} E_j(z)dz\right| \leq \sum_j \left[\varepsilon(4\sqrt{2})s^2 + \varepsilon s\sqrt{2} \cdot \ell(\Gamma \cap Q_j)\right].$$

For the first piece, we note that s^2 is the area of a small square, so

$$\sum_j \varepsilon(4\sqrt{2})s^2 \leq \varepsilon(4\sqrt{2}) \cdot (\text{area of large square } Q).$$

For the second piece, we note that the lengths $\ell(\Gamma \cap Q_j)$ add up to the total length $\ell(\Gamma)$. Also we can certainly assume that $s \leq 1$. Therefore,

$$\sum_j \varepsilon s\sqrt{2} \cdot \ell(\Gamma \cap Q_j) \leq \varepsilon\sqrt{2} \cdot \ell(\Gamma).$$

In total,

$$\left| \int_\Gamma f(z)dz \right| \leq \varepsilon \left[(4\sqrt{2}) \cdot (\text{area of } Q) + \sqrt{2} \cdot \ell(\Gamma) \right],$$

which is ε multiplied by a fixed constant. By taking ε to be arbitrarily small, the absolute value $|\int_\Gamma f(z)dz|$ is less than every positive number, so it must be zero. □