Math 4220: Final Practice Exam

Solutions

1. Solve for z: $\sin(z) = \dfrac{e^{iz} - e^{-iz}}{2i} = 0$.

This means $e^{iz} = e^{-iz}$. Write $z = x + iy$, then

$e^{i(x+iy)} = e^{-i(x+iy)} \Rightarrow e^{-y+ix} = e^{y-ix}$

$\Rightarrow e^{-y} e^{ix} = e^{y} e^{i(-x)}$. This means $e^{-y} = e^{y}$ and

$x = -x + 2k\pi$ for some integer k. So $-y = y$, $y = 0$

and $2x = 2k\pi$, $x = k\pi$. Therefore $z = x + iy = k\pi$.

2. ⓐ If $f = u + iv$, then $\dfrac{df}{dz} = \dfrac{\partial f}{\partial x} = \dfrac{\partial f}{\partial (iy)}$

$= \left[\dfrac{\partial u}{\partial x} + i \dfrac{\partial v}{\partial x}\right]$ and also $= \dfrac{1}{i}\left[\dfrac{\partial u}{\partial y} + i \dfrac{\partial v}{\partial y}\right] = \dfrac{\partial v}{\partial y} - i \dfrac{\partial u}{\partial y}$.

This means in particular that $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ and $\dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y}$

(the Cauchy-Riemann equations, which we already knew), and

$\dfrac{df}{dz} = \dfrac{\partial u}{\partial x} - i \dfrac{\partial u}{\partial y}$.

ⓑ It is enough to check the Cauchy-Riemann equations for

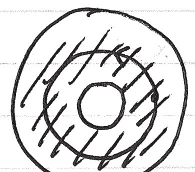
g: we must verify that $\dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial x}\right) = \dfrac{\partial}{\partial y}\left(-\dfrac{\partial u}{\partial y}\right)$

and that $\dfrac{\partial}{\partial y}\left(\dfrac{\partial u}{\partial x}\right) = -\dfrac{\partial}{\partial x}\left(-\dfrac{\partial u}{\partial y}\right)$.

The first equation is true because u is harmonic,

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$. The second equation is true by equality of mixed partials.

(c) Since the domain D is simply connected, every simple closed curve in D has the property that the interior of the curve is entirely contained in D. (Note, this would not be true for something like an annulus: in



this picture, the curve is in D but its interior is not.) Therefore, by Cauchy's Thm., $\int_\gamma g(z)\,dz = 0$ for every simple closed curve γ in D, hence g has an antiderivative on D by the path independence theorem.

(d) Write $G = U + iV$, then $w = U - u$. We know $\frac{dG}{dz} = g$, therefore $\frac{\partial U}{\partial x} - i\frac{\partial U}{\partial y} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$. (The left side $\frac{dG}{dz} = \frac{\partial U}{\partial x} - i\frac{\partial U}{\partial y}$ follows from the argument in (a), using G in place of f. The right side $g = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$ is the definition of g.) It follows that

$\frac{\partial w}{\partial x} - i\frac{\partial w}{\partial y} = 0$, thus $\frac{\partial w}{\partial x} = 0$ and $\frac{\partial w}{\partial y} = 0$, so

w is a constant function. Say $w(x,y) = c$, then

$U - u = c$, so $u = U - c$. The function $f = G - c$

$= (U - c) + iV$ is analytic on D and has real part u.

3. $$\left|\sin(x+iy)\right|^2 = \sin(x+iy)\cdot\overline{\sin(x+iy)}$$

$$= \frac{e^{i(x+iy)} - e^{-i(x+iy)}}{2i}\cdot\frac{e^{-i(x-iy)} - e^{i(x-iy)}}{-2i}$$

$$= \frac{1}{4}\left[e^{-y+ix} - e^{y-ix}\right]\left[e^{-y-ix} - e^{y+ix}\right]$$

$$= \frac{1}{4}\left[e^{-2y} - e^{2ix} - e^{-2ix} + e^{2y}\right]$$

$$= \frac{1}{2}\left[\frac{e^{2y}+e^{-2y}}{2} - \frac{e^{2ix}+e^{-2ix}}{2}\right]$$

$$= \frac{1}{2}\left[\cosh(2y) - \cos(2x)\right].$$

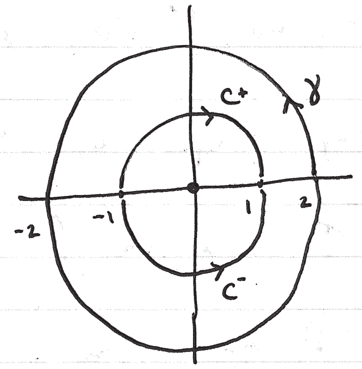
4. $$z^{1/n} = e^{\frac{1}{n}\log z} = e^{\frac{1}{n}[\ln(r) + i(\theta + 2k\pi)]} \quad (k \text{ can be any integer})$$

$$= e^{\frac{1}{n}\ln(r)}\cdot e^{i\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right)} = r^{\frac{1}{n}} e^{i\left(\frac{\theta}{n}+\frac{2k\pi}{n}\right)}.$$

Note that values of k outside $0, 1, \dots, n-1$ lead to repeat values of $e^{i\left(\frac{\theta}{n}+\frac{2k\pi}{n}\right)}$, since if $k' = k + cn$ for some integer c,

$$e^{i\left(\frac{\theta}{n}+\frac{2k'\pi}{n}\right)} = e^{i\left(\frac{\theta}{n}+\frac{2k\pi}{n}+2\pi c\right)} = e^{i\left(\frac{\theta}{n}+\frac{2k\pi}{n}\right)}.$$

5.



ⓐ The counterclockwise circle centered at 0 of radius 1 is $-C^+ + C^-$, therefore

$$\int_{-C^+ + C^-} f(z)\,dz = -3 + 7 = 4.$$

Since γ can be deformed to this circle by a continuous deformation that avoids the singularity at 0, by Cauchy's theorem, $\int_\gamma f(z)\,dz = 4$ also.

Alternatively, $\int_{-C^+ + C^-} f(z)\,dz = 2\pi i \operatorname{Res}(f; 0) = \int_\gamma f(z)\,dz.$

ⓑ We require that the integral of g around any positively oriented simple closed curve equal 0. Either the curve encloses 0, in which case the integral is $2\pi i \operatorname{Res}(f - \frac{c}{z}; 0)$, or it does not enclose 0, in which case the integral is zero automatically. So it is enough to find c such that $\operatorname{Res}(f - \frac{c}{z}; 0) = 0$. We know from ⓐ that $2\pi i \operatorname{Res}(f; 0) = 4$, thus $\operatorname{Res}(f; 0) = \frac{2}{\pi i}$. Thus

$\mathrm{Res}(f - \frac{c}{z}; 0) = \mathrm{Res}(f; 0) - \mathrm{Res}(\frac{c}{z}; 0) = \frac{2}{\pi i} - c = 0,$

so $c = \frac{2}{\pi i}$ works.

6. Since f' is analytic, it satisfies the mean value property: $f'(0) = \frac{1}{2\pi} \int_0^{2\pi} f'(re^{it})\, dt$. Therefore it suffices to show that also $f'(0) = \frac{1}{2\pi} \int_0^{2\pi} \frac{f(re^{it})}{re^{it}}\, dt$.

If C is the circle of radius r centered at 0, oriented counterclockwise, the Cauchy integral formula (generalized version) says that $f'(0) = \frac{1}{2\pi i} \int_C \frac{f(w)}{(w-0)^2}\, dw$.

Parametrize C by $w(t) = re^{it}$, $0 \le t \le 2\pi$:

$$f'(0) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(re^{it})}{(re^{it})^2} \cdot ire^{it}\, dt = \frac{1}{2\pi} \int_0^{2\pi} \frac{f(re^{it})}{re^{it}}\, dt.$$

7. If $f(z) - \frac{c}{z-z_0}$ has a removable singularity at z_0, the Laurent series for $f(z) - \frac{c}{z-z_0}$ ~~[illegible]~~ in a neighborhood of z_0 has all non-negative terms:

$$f(z) - \frac{c}{z-z_0} = \sum_{n=0}^{\infty} a_n (z-z_0)^n.$$ (Note the lower limit $n=0$.)

Thus $f(z) = \frac{c}{z-z_0} + \sum_{n=0}^{\infty} a_n (z-z_0)^n$, so $\operatorname{Res}(f; z_0) = c$

by definition.

For the converse, any function with a pole of order at least two at z_0 will work. Say $f(z) = \frac{1}{(z-z_0)^2}$, then $\operatorname{Res}(f; z_0) = 0$, but $f(z) - \frac{0}{z-z_0}$ does not have a removable singularity at z_0. In fact, the statement is "if and only if" under the extra assumption that the pole of f at z_0 is simple.

8. ⓐ We have $\lim_{z\to\infty} f(z) = 8$. Say $f(z) = \sum_{n=-\infty}^{\infty} c_n z^n$ for $|z|>1$. If there existed $k>0$ such that $c_k \neq 0$, we would have $\lim_{z\to\infty} f(z) = \lim_{z\to 0}\left(\sum_{n=-\infty}^{\infty} c_n z^n\right) = \infty$

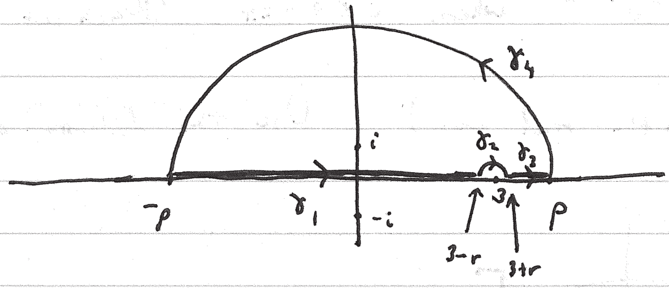
(or maybe the limit might not exist, but in any case it could not equal a constant). Therefore we may write

$$f(z) = \sum_{n=-\infty}^{0} c_n z^n \quad \text{for } |z|>1, \text{ so } \lim_{z\to\infty} f(z) = c_0 = 8.$$

ⓑ Since $|z|>1$, $\frac{8z}{z-1} = \frac{8}{1-1/z} = 8\left(1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right)$

$$= \sum_{n=-\infty}^{0} 8z^n, \text{ or } \sum_{n=0}^{\infty} \frac{8}{z^n}.$$

9. Let $f(z) = \frac{1}{(z^2+1)(z-3)}$, poles at $\pm i$ and 3. Use the following contour:



We know: $\displaystyle\int_{\gamma_1+\gamma_2+\gamma_3+\gamma_4} f(z)\,dz = 2\pi i\cdot \mathrm{Res}(f;i)$,

$\displaystyle\lim_{\rho\to\infty}\int_{\gamma_4} f(z)\,dz = 0$ (since f is a rational function where the degree of the denominator $\geq$ degree of numerator $+2$),

$$\lim_{\substack{\rho\to\infty\\ r\to 0}}\int_{\gamma_1+\gamma_3} f(z)\,dz = \text{p.v.}\int_{-\infty}^{\infty}\frac{dx}{(x^2+1)(x-3)},$$

and $\displaystyle\lim_{r\to 0}\int_{\gamma_2} f(z)\,dz = -\pi i\,\mathrm{Res}(f;3)$ (since 3 is a simple pole).

Therefore,

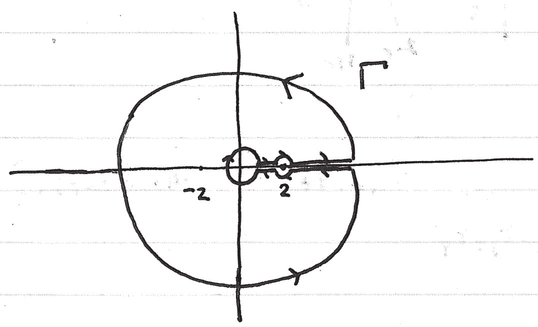
$$\text{p.v.}\int_{-\infty}^{\infty}\frac{dx}{(x^2+1)(x-3)} = 2\pi i\,\mathrm{Res}(f;i) + \pi i\,\mathrm{Res}(f;3)$$

$$= 2\pi i\cdot\frac{1}{(x+i)(x-3)}\Big|_{x=i} + \pi i\cdot\frac{1}{x^2+1}\Big|_{x=3} = \frac{2\pi i}{2i(-3+i)} + \frac{\pi i}{10}$$

$$= \frac{\pi}{-3+i} + \frac{\pi i}{10} = \pi\cdot\frac{-3-i}{(-3+i)(-3-i)} + \frac{\pi i}{10} = \pi\cdot\frac{-3-i}{10} + \frac{\pi i}{10} = \frac{-3\pi}{10}.$$

10. Let $f(z) = \frac{1}{z^{1/2}(z^2-4)}$ using the branch of $z^{1/2}$ given by

$(re^{i\theta})^{1/2} = r^{1/2}e^{i\theta/2}$ where $0 \le \theta < 2\pi$. (Branch cut is along the positive real axis.) Use the following contour:



Let ε be the radius of the small circle centered at 0,
r be the radius of the circle centered at 2,
ρ be the radius of the large circle centered at 0.

We know $\int_\Gamma f(z)dz = 2\pi i \operatorname{Res}(f, -2)$.

As $\varepsilon \to 0$, the integral around the inner circle tends to 0

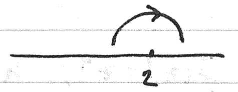
since it is bounded by $(2\pi\varepsilon) \cdot \frac{\text{Constant}}{\varepsilon^{1/2}} = \text{Constant} \cdot \varepsilon^{1/2}$.

(The dominant term in the denominator $z^{5/2} - 4z^{1/2}$ is $4z^{1/2}$.)

As $\rho \to \infty$, the integral around the outer circle tends to 0

since it is bounded by $(2\pi\rho) \cdot \frac{\text{Constant}}{\rho^{5/2}}$. (The dominant term in the denominator $z^{5/2} - 4z^{1/2}$ is $z^{5/2}$.)

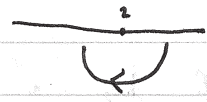
To compute the integrals along the small circle centered at 2, we can't use $\mathrm{Res}(f;2)$ since f is not analytic at 2.

However, for the upper semicircle, 

we can say that f behaves like its upper branch, so the integral converges to $-\pi i \cdot \dfrac{1}{2^{1/2}(2+2)}$.

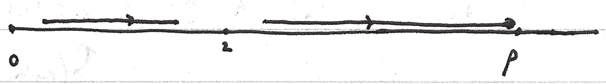
"upper value"

For the lower semicircle, f behaves like its lower branch, so

the integral  converges to $-\pi i \cdot \dfrac{1}{2^{1/2}e^{i\pi}(2+2)} = \pi i \cdot \dfrac{1}{2^{1/2}\cdot 4}$.

"lower value"

Finally, the horizontal portions: The upper horizontal

integrals 

converge to $\mathrm{p.v.}\displaystyle\int_0^\infty \frac{1}{x^{1/2}(x^2-4)}\,dx$, and the lower horizontal

integrals converge to $-\mathrm{p.v.}\displaystyle\int_0^\infty \frac{1}{x^{1/2}e^{i\pi}(x^2-4)}\,dx = \mathrm{p.v.}\int_0^\infty \frac{1}{x^{1/2}(x^2-4)}\,dx.$

Putting it all together,

$$2\,\mathrm{p.v.}\int_0^\infty \frac{dx}{x^{1/2}(x^2-4)} - \pi i\cdot\frac{1}{2^{1/2}\cdot 4} + \pi i\cdot\frac{1}{2^{1/2}\cdot 4} = 2\pi i\,\mathrm{Res}(f;-2)$$

$$= 2\pi i\cdot\frac{1}{2^{1/2}e^{i\pi/2}(-2-2)} = \frac{2\pi i}{2^{1/2}\cdot i(-4)} = \frac{-\pi}{2\sqrt{2}}.$$

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We conclude that $\text{p.v.}\displaystyle\int_0^\infty \frac{dx}{\sqrt{x}\,(x^2-4)} = \frac{-\pi}{4\sqrt{2}}$.

11. Use $f(z) = z^4$, $h(z) = 3z^3 - 2z + 7$. If $|h(z)| < |f(z)|$ for all z on the circle of radius R centered at 0, then Rouché's Thm. implies that f and $f+h$ have the same number of roots in the disk $\{|z| < R\}$, counting multiplicity. This is 4 roots because f has a zero of order 4 at the origin. So we seek R such that $|h(z)| < R^4$ whenever $|z| = R$. We know $|h(z)| \le 3|z|^3 + 2|z| + 7 = 3R^3 + 2R + 7$.

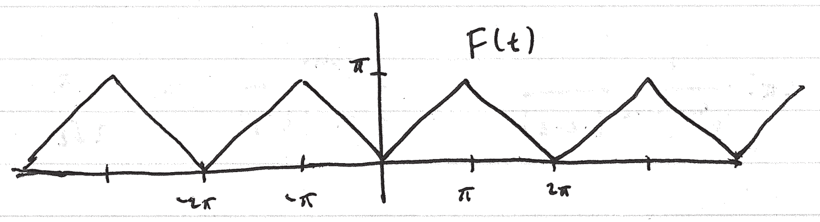
(note the + sign!)

Therefore, if $3R^3 + 2R + 7 < R^4$, this value of R works.

Take $R = 4$, then the left side is $3 \cdot 64 + 8 + 7 < 256$.

(In fact, Wolfram Alpha says that the zeros of the polynomial with highest modulus are $-2.24 \pm 0.55i$, with modulus 2.31.)

12.



ⓐ $c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(t)e^{-int}\,dt$. If $n=0$,

$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(t)\,dt = \frac{1}{2\pi}(\pi^2) = \frac{\pi}{2}.$$

If $n\neq 0$, $c_n = \frac{1}{2\pi}\int_{-\pi}^{0} -te^{-int}\,dt + \frac{1}{2\pi}\int_{0}^{\pi} te^{-int}\,dt$

$$= \frac{1}{2\pi}\left[(-t)\cdot\frac{e^{-int}}{-in}\Big|_{t=-\pi}^{0} - \int_{-\pi}^{0}(-1)\frac{e^{-int}}{-in}\,dt\right]$$

$$+ \frac{1}{2\pi}\left[t\cdot\frac{e^{-int}}{-in}\Big|_{t=0}^{\pi} - \int_{0}^{\pi}\frac{e^{-int}}{-in}\,dt\right]$$

$$= \frac{1}{2\pi}\left[\frac{+\pi}{in}e^{in\pi} - \frac{1}{in}\cdot\frac{e^{-int}}{-in}\Big|_{t=-\pi}^{0}\right] + \frac{1}{2\pi}\left[\frac{-\pi}{in}e^{-in\pi} + \frac{1}{in}\cdot\frac{e^{-int}}{-in}\Big|_{t=0}^{\pi}\right]$$

$$= \frac{1}{2\pi}\left[\frac{\pi}{in}e^{in\pi} - \frac{1}{n^2}\left(1 - e^{in\pi}\right)\right] + \frac{1}{2\pi}\left[\frac{-\pi}{in}e^{-in\pi} + \frac{1}{n^2}\left(e^{-in\pi} - 1\right)\right].$$

When n is even, $e^{in\pi} = e^{-in\pi} = 1$, so we get

$$c_n = \frac{1}{2\pi}\left[\frac{\pi}{in}\right] + \frac{1}{2\pi}\left[-\frac{\pi}{in}\right] = 0.$$

When n is odd, $e^{in\pi} = e^{-in\pi} = -1$, so we get

$$c_n = \frac{1}{2\pi}\left[-\frac{\pi}{in} - \frac{2}{n^2}\right] + \frac{1}{2\pi}\left[\frac{\pi}{in} - \frac{2}{n^2}\right] = \frac{-2}{\pi n^2}.$$

Therefore $c_n = \begin{cases} \pi/2, & n=0 \\ 0, & n \text{ even}, n\neq 0 \\ -2/\pi n^2, & n \text{ odd}. \end{cases}$

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(b) First solve $f'(t) + 4f(t) = e^{int}$. If

$f(t) = ae^{int}$ then $a \cdot in e^{int} + 4ae^{int} = e^{int}$,

$a(in+4) = 1$, $a = \frac{1}{in+4}$.

Therefore a solution to $f'(t) + 4f(t) = \sum_{n=-\infty}^{\infty} c_n e^{int}$

is $f(t) = \sum_{n=-\infty}^{\infty} \frac{c_n}{in+4} e^{int}$

$$= \frac{1}{4} \cdot \frac{\pi}{2} + \sum_{n \text{ odd}} \frac{1}{in+4} \cdot \frac{-2}{\pi n^2} e^{int}.$$

13. (a) The Fourier transform of $F(t)$ is

$$G(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(t) e^{-i\omega t} dt = \frac{1}{2\pi} \int_0^{\infty} e^{-t} e^{-i\omega t} dt$$

$$= \frac{1}{2\pi} \int_0^{\infty} e^{-(1+i\omega)t} dt = \frac{1}{2\pi} \cdot \left. \frac{e^{-(1+i\omega)t}}{-(1+i\omega)} \right|_{t=0}^{\infty}$$

$$= \frac{1}{2\pi} \cdot \frac{-1}{1+i\omega} (0-1) = \frac{1}{2\pi(1+i\omega)}.$$

(b) $f'(t) + 4f(t) = F(t)$, Fourier transform

of both sides gives $i\omega g(\omega) + 4g(\omega) = \frac{1}{2\pi(1+i\omega)}$.

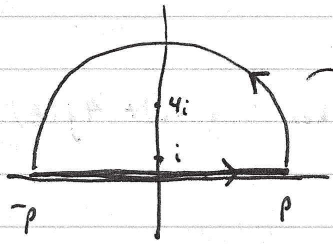
Therefore, $g(\omega) = \frac{1}{2\pi(4+i\omega)(1+i\omega)}$.

(c) $$f(t) = \text{p.v.}\int_{-\infty}^{\infty} g(\omega)e^{i\omega t}\,d\omega = \frac{1}{2\pi}\,\text{p.v.}\int_{-\infty}^{\infty} \frac{1}{(4+i\omega)(1+i\omega)}\,e^{i\omega t}\,d\omega.$$

Poles at $4+i\omega=0,\quad 1+i\omega=0$

$\omega = 4i,\quad \omega = i.$

When $t>0$ use an upper semicircular contour:



integral along the semicircle tends to 0 as $\rho\to\infty$ by Jordan's Lemma

Therefore, $$\frac{1}{2\pi}\,\text{p.v.}\int_{-\infty}^{\infty} g(\omega)e^{i\omega t}\,d\omega = \frac{1}{2\pi}\cdot 2\pi i\left[\text{Res}\left(g(\omega)e^{i\omega t};\, i\right) + \text{Res}\left(g(\omega)e^{i\omega t};\, 4i\right)\right]$$

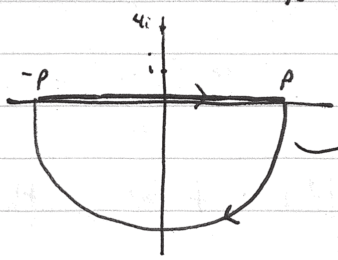
$$= i\cdot\left[\frac{\omega-i}{(4+i\omega)(1+i\omega)}\,e^{i\omega t}\Big|_{\omega=i} + \frac{\omega-4i}{(4+i\omega)(1+i\omega)}\,e^{i\omega t}\Big|_{\omega=4i}\right].$$

Note, $\dfrac{\omega-i}{1+i\omega} = -i,\quad \dfrac{\omega-4i}{4+i\omega} = -i.$ So,

$$f(t) = i\cdot\left[\frac{-i}{3}e^{-t} + \frac{-i}{-3}e^{-4t}\right] = \frac{1}{3}e^{-t} - \frac{1}{3}e^{-4t}$$

when $t>0$.

When $t\le 0$ use a lower semicircular contour:



integral along the semicircle tends to 0 as $\rho\to\infty$ by Jordan's Lemma when $t<0$, and since the denominator of $\dfrac{1}{(4+i\omega)(1+i\omega)}$ has degree 2 when $t=0$.

Since there are no poles inside the contour,

$$f(t) = \frac{1}{2\pi} \text{p.v.} \int_{-\infty}^{\infty} g(\omega) e^{i\omega t}\, dt \quad = 0 \quad \text{when } t \leq 0.$$

Conclusion: $f(t) = \begin{cases} 0, & t \leq 0 \\ \frac{1}{3} e^{-t} - \frac{1}{3} e^{-4t}, & t > 0. \end{cases}$

(d) If $g(t) = Ae^{-4t}$, then $g'(t) + 4g(t) = -4Ae^{-4t} + 4Ae^{-4t}$

which equals 0. ✓

Solution to initial-value problem:

$$f(t) = \underbrace{\frac{1}{3} e^{-t} - \frac{1}{3} e^{-4t}}_{\substack{\text{particular solution} \\ \text{to original diff. eq.}}} + \underbrace{Ae^{-4t}}_{\substack{\text{general solution to} \\ \text{homogeneous diff. eq.}}}.$$

Solve for A: $f(0) = 2 = \frac{1}{3} - \frac{1}{3} + A$, so $A = 2$.

Therefore $f(t) = \frac{1}{3} e^{-t} + \frac{5}{3} e^{-4t}$.

(e) Initial-value problem: $f'(t) + 4f(t) = e^{-t}$ $(t \geq 0)$,
$f(0) = 2$.

Laplace transform of both sides:

$$\mathcal{L}\{f'\}(s) + 4\mathcal{L}\{f\}(s) = \mathcal{L}\{e^{-t}\} = \frac{1}{s+1}.$$

We know $\mathcal{L}\{f'\}(s) = s\mathcal{L}\{f\}(s) - f(0)$, therefore

$$s\mathcal{L}\{f\}(s) - 2 + 4\mathcal{L}\{f\}(s) = \frac{1}{s+1}$$

$$(s+4)\mathcal{L}\{f\}(s) = 2 + \frac{1}{s+1}$$

$$\mathcal{L}\{f\}(s) = \frac{2}{s+4} + \frac{1}{(s+1)(s+4)}.$$

Partial fractions on $\frac{1}{(s+1)(s+4)}$: it equals $\frac{A}{s+1} + \frac{B}{s+4}$

where $A = \left.\frac{s+1}{(s+1)(s+4)}\right|_{s=-1} = \frac{1}{3}$, $B = \left.\frac{s+4}{(s+1)(s+4)}\right|_{s=-4} = -\frac{1}{3}$.

Thus $\mathcal{L}\{f\}(s) = \frac{2}{s+4} + \frac{1/3}{s+1} - \frac{1/3}{s+4} = \frac{1/3}{s+1} + \frac{5/3}{s+4}$.

We know that the Laplace transform of $\frac{1}{3}e^{-t}$ is $\frac{1/3}{s+1}$,

and the Laplace transform of $\frac{5}{3}e^{-4t}$ is $\frac{5/3}{s+4}$. Therefore

without needing the inversion formula, we can conclude

that $f(t) = \frac{1}{3}e^{-t} + \frac{5}{3}e^{-4t}$.