Math 4220: Final Exam

December 11, 2015

Name:

Instructions: This exam is 150 minutes long. It has 14 pages including the cover and 9 questions worth a total of 100 points. No written or electronic aids are allowed.

Please fully explain all your answers. If you need more space to answer a question, use the back of the *preceding* sheet or the blank sheet at the end of the exam. Label your work clearly if you use extra space.

Academic integrity is expected of all Cornell University students at all times, whether in the presence or absence of members of the faculty. Understanding this, I declare I shall not give, use, or receive unauthorized aid in this examination.

Please sign below to indicate that you have read and agree to these instructions.

Signature:

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1. (10 points) (a) Let $f(z)$ be an analytic function such that $f(1+2i) = 4-i$ and $f'(1+2i) = 2+3i$. Write the linear approximation for $f(z)$ near $1+2i$.

$$f(z) \approx f(z_0) + f'(z_0)(z-z_0)$$

$$= (4-i) + (2+3i)(z-(1+2i))$$

when $z_0 = 1+2i$.

(b) If $f(x+iy) = u(x,y) + iv(x,y)$, write the linear approximations for u and v near $(1,2)$. Use these to compute the partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ at $(1,2)$, and confirm that the Cauchy-Riemann equations hold.

$$u + iv \approx (4-i) + (2+3i)((x+iy) - (1+2i))$$

$$= 4-i + (2+3i)((x-1) + i(y-2))$$

$$= 4-i + 2(x-1) - 3(y-2) + i(3(x-1) + 2(y-2))$$

$$\Rightarrow \quad u(x,y) \approx 4 + 2(x-1) - 3(y-2)$$

$$v(x,y) \approx -1 + 3(x-1) + 2(y-2)$$

At $(1,2)$: $\quad \frac{\partial u}{\partial x} = 2, \quad \frac{\partial u}{\partial y} = -3, \quad \frac{\partial v}{\partial x} = 3, \quad \frac{\partial v}{\partial y} = 2$

Cauchy-Riemann equations: $\quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ ✓

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \quad ✓$$

2. (10 points) (a) What are the three values of $(-8)^{1/3}$? Express your answers in $re^{i\theta}$ form.

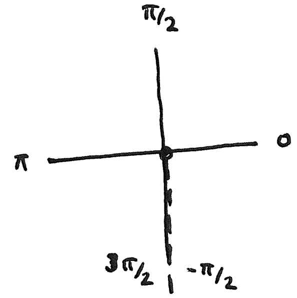
$$z^{\frac{1}{3}} = |z|^{1/3} \cdot e^{i\left(\frac{1}{3}\arg(z)\right)}$$

$$\Rightarrow (-8)^{1/3} = 2 \cdot e^{i\left(\frac{1}{3}(\pi + 2k\pi)\right)}$$

$$= 2e^{i\pi/3}, \quad 2e^{i\pi}, \quad 2e^{i(5\pi/3)}.$$

(b) Write an explicit formula for a function $f(z)$ which is a branch of $z^{1/3}$ where the branch cut is along the negative imaginary axis. You may use the notation $\arg_\theta(z)$ to denote the argument of z in the interval $(\theta, \theta + 2\pi]$. Compute $f(-8)$ and $f(8)$.

$$f(z) = |z|^{1/3} \cdot e^{i \cdot \frac{1}{3}\arg_{-\pi/2}(z)}$$

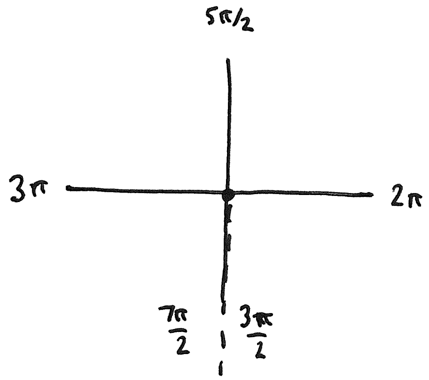


$$f(-8) = 2e^{i\pi/3}$$

$$f(8) = 2e^{i \cdot 0} = 2$$

(c) Write an explicit formula for another function $g(z)$ which is also a branch of $z^{1/3}$ where the branch cut is along the negative imaginary axis, but for which $g(-8) \neq f(-8)$. Compute $g(-8)$ and $g(8)$.

$$g(z) = |z|^{1/3} \cdot e^{i \cdot \frac{1}{3} \arg_{3\pi/2}(z)}$$



$$g(-8) = 2e^{i\pi} = -2$$

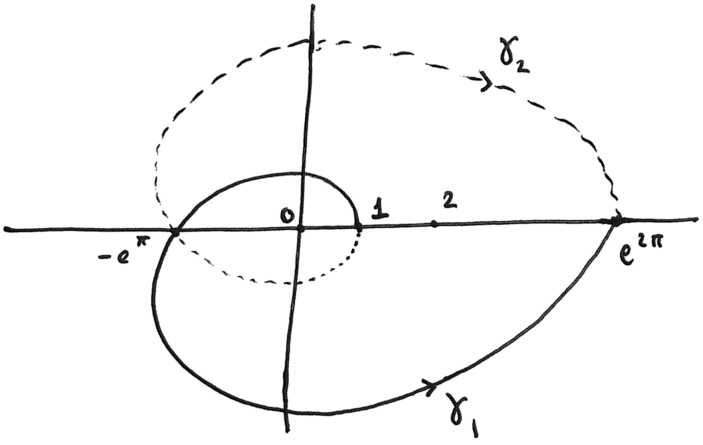
$$g(8) = 2e^{i \cdot 2\pi/3}$$

(Another choice would be $g(z) = |z|^{1/3} \cdot e^{i \cdot \frac{1}{3} \arg_{7\pi/2}(z)}$.)

After that, it repeats.

3. (10 points) (a) Let γ_1 be the curve parametrized by $z(t) = e^{(1+i)t}$, $0 \le t \le 2\pi$. Let γ_2 be the curve parametrized by $z(t) = e^{(1-i)t}$, $0 \le t \le 2\pi$. Draw both curves γ_1 and γ_2 on the same graph, labeling the endpoints and the orientations.

$\gamma_1 : \ z(t) = e^{t} e^{it}$, $\quad \gamma_2 : \ z(t) = e^{t} e^{-it}$



(b) Compute $\displaystyle\int_{\gamma_1-\gamma_2} \left(\frac{1}{z} + \frac{e^z}{2-z}\right) dz.$

$\gamma_1 - \gamma_2$ is a closed curve that goes around $z=0$ twice counterclockwise and $z=2$ once counterclockwise.

Therefore, $\displaystyle\int_{\gamma_1-\gamma_2} \frac{1}{z}\, dz = 2 \cdot 2\pi i$ and

$$\int_{\gamma_1-\gamma_2} \frac{e^z}{2-z}\, dz = 2\pi i \operatorname{Res}\left(\frac{e^z}{2-z} ; 2\right) = 2\pi i \left(-e^2\right).$$

So, $\displaystyle\int_{\gamma_1-\gamma_2} \left(\frac{1}{z} + \frac{e^z}{2-z}\right) dz = 4\pi i - 2e^2\pi i = (4-2e^2)\pi i.$

4. (10 points) (a) Suppose $f(z)$ is analytic in a neighborhood of the origin, and $f(z) - \cos(z)$ has a zero of order 3 at the origin. What can be said about the values $f(0)$, $f'(0)$, $f''(0)$, $f'''(0)$?

Let the Taylor series of $f(z)$ centered at 0 be

$$f(z) = \sum_{n=0}^{\infty} c_n z^n , \quad c_n = \frac{f^{(n)}(0)}{n!} .$$ Then

$$f(z) - \cos(z) = (c_0 - 1) + c_1 z + (c_2 + \tfrac{1}{2}) z^2 + c_3 z^3 + \cdots .$$

This function has a zero of order 3 at the origin, so

$$c_0 - 1 = 0 , \quad c_1 = 0 , \quad c_2 + \tfrac{1}{2} = 0 , \quad c_3 \neq 0 .$$

Hence $f(0) = 1$, $f'(0) = 0$, $f''(0) = -1$, $f'''(0) \neq 0$.

(b) Let $g(z) = \dfrac{1}{f'(z) + z}$. Does g have a singularity at the origin? If so, classify it as a removable singularity, a pole whose order you specify, or an essential singularity. Justify your answer.

$$f(z) = 1 - \tfrac{1}{2} z^2 + c_3 z^3 + c_4 z^4 + \cdots ,$$ so

$$f'(z) + z = 0 + 0z + 3c_3 z^2 + 4c_4 z^3 + \cdots$$

has a zero of order 2 at the origin since $c_3 \neq 0$.

Therefore, $g(z)$ has a pole of order 2 at the origin.

5. (10 points) Suppose $f(z)$ is analytic on the annulus $A = \{r < |z| < R\}$. Prove that there exist functions $g(z)$ and $h(z)$ such that g is analytic on the disk $\{|z| < R\}$, h is analytic on the region $\{|z| > r\}$, and $f(z) = g(z) + h(z)$ for all z in A.

Let $f(z) = \sum_{n=-\infty}^{\infty} c_n z^n$ be the Laurent series for f centered at 0 that converges on A.

Define $g(z) = \sum_{n=0}^{\infty} c_n z^n$, $h(z) = \sum_{n=-\infty}^{-1} c_n z^n$. Both g and h are analytic wherever they converge, and on A, both g and h converge with $f = g + h$.

As g is a power series centered at 0, it must have a radius of convergence, which is at least R since g converges on A. Thus g converges on the disk $\{|z| < R\}$.

As h is a power series in the variable $1/z$, it converges in a region $\{z : |\frac{1}{z}| < b\}$ (and maybe also parts of the boundary circle). This is $\{z : |z| > \frac{1}{b}\}$, and since h converges on A, we know $\frac{1}{b} \leq r$, so h converges on $\{|z| > r\}$.

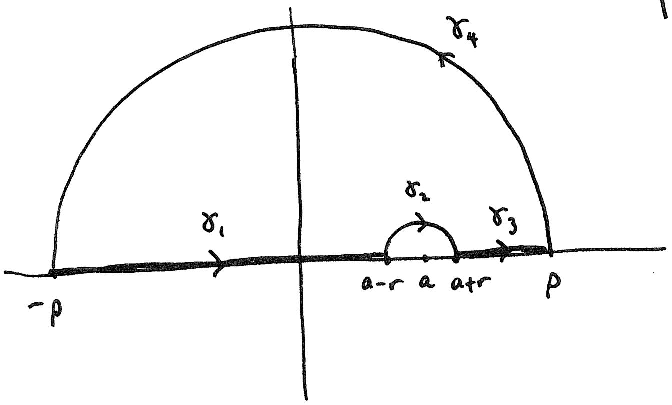
6. (12 points) For fixed $a \in \mathbf{R}$, compute p.v. $\int_{-\infty}^{\infty} \frac{\cos(t)}{t-a}\,dt$.

Hint: Use $\cos(t) = \mathrm{Re}(e^{it})$.

$$\text{p.v.} \int_{-\infty}^{\infty} \frac{\cos(t)}{t-a}\,dt = \text{p.v.} \int_{-\infty}^{\infty} \frac{\mathrm{Re}(e^{it})}{t-a}\,dt = \mathrm{Re}\left[\text{p.v.} \int_{-\infty}^{\infty} \frac{e^{it}}{t-a}\,dt\right].$$

Let $f(z) = \frac{e^{iz}}{z-a}$. Use this contour:

$\Gamma = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$



Since f is analytic on and inside Γ, $\int_{\Gamma} f(z)\,dz = 0$.

Also, $\lim_{p \to \infty} \int_{\gamma_4} f(z)\,dz = 0$ (Jordan's Lemma),

$\lim_{r \to 0} \int_{\gamma_2} f(z)\,dz = -i\pi \,\mathrm{Res}(f; a) = -i\pi e^{ia}$,

$\lim_{\substack{p \to \infty \\ r \to 0}} \int_{\gamma_1 + \gamma_3} f(z)\,dz = \text{p.v.} \int_{-\infty}^{\infty} \frac{e^{it}}{t-a}\,dt$. Therefore,

$$\text{p.v.} \int_{-\infty}^{\infty} \frac{e^{it}}{t-a}\,dt - i\pi e^{ia} = 0 \Rightarrow \text{p.v.} \int_{-\infty}^{\infty} \frac{\cos(t)}{t-a}\,dt = \mathrm{Re}\left[i\pi e^{ia}\right]$$

$$= \mathrm{Re}\left[i\pi \cos(a) - \pi \sin(a)\right] = -\pi \sin(a).$$

7. (10 points) (a) Find constants c_n such that $\sin(t) + \cos(3t) = \sum_{n=-\infty}^{\infty} c_n e^{int}$.

$$\sin(t) + \cos(3t) = \frac{e^{it} - e^{-it}}{2i} + \frac{e^{3it} + e^{-3it}}{2}$$

so, $c_1 = \frac{1}{2i}$, $c_{-1} = -\frac{1}{2i}$, $c_3 = c_{-3} = \frac{1}{2}$, $c_n = 0$ for all other n.

(b) Suppose c_n is given. Find a solution of the form $f(t) = A_n e^{int}$ to the differential equation $f'(t) + 5f(t) = c_n e^{int}$.

$$f(t) = A_n e^{int}, \quad f'(t) = in A_n e^{int}$$

$in A_n e^{int} + 5A_n e^{int} = c_n e^{int}$ will be true when

$in A_n + 5A_n = c_n$, $A_n = \frac{c_n}{5 + in}$. Solution: $f(t) = \frac{c_n}{5+in} e^{int}$.

(c) Use parts (a) and (b) to find a periodic solution to the differential equation $f'(t) + 5f(t) = \sin(t) + \cos(3t)$. You do not need to simplify your answer.

$$f'(t) + 5f(t) = c_1 e^{it} + c_{-1} e^{-it} + c_3 e^{3it} + c_{-3} e^{-3it}$$

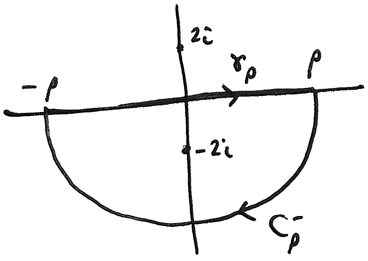
$$f(t) = \frac{c_1}{5+i} e^{it} + \frac{c_{-1}}{5-i} e^{-it} + \frac{c_3}{5+3i} e^{3it} + \frac{c_{-3}}{5-3i} e^{-3it}$$

$$= \frac{1}{2i(5+i)} e^{it} - \frac{1}{2i(5-i)} e^{-it} + \frac{1}{2(5+3i)} e^{3it} + \frac{1}{2(5-3i)} e^{-3it}.$$

8. (18 points) (a) Let $F(t) = \dfrac{1}{t^2+4}$ and let $G(\omega)$ be its Fourier transform. By evaluating the formula for G as a principal value integral, show that $G(\omega) = \frac{1}{4}e^{-2\omega}$ when $\omega \geq 0$.

$$G(\omega) = \frac{1}{2\pi}\ \text{p.v.}\int_{-\infty}^{\infty} \frac{1}{t^2+4}\, e^{-i\omega t}\, dt .$$

Let $f(z) = \dfrac{1}{z^2+4}\, e^{-i\omega z}$. Use contour $\Gamma = \gamma_\rho + C_\rho^-$:



Since $\omega \geq 0$, on C_ρ^-, $\left|e^{-i\omega z}\right| \leq 1$,

therefore

$$\lim_{\rho\to\infty}\int_{C_\rho^-}|f(z)|dz \leq \lim_{\rho\to\infty}\int_{C_\rho^-}\left|\frac{1}{z^2+4}\right| dz = 0$$

Since $\dfrac{1}{z^2+4}$ is a rational function with degree (denominator) $\geq 2 +$ degree (numerator). (Note: Jordan's Lemma would also work here, but only when $\omega > 0$.) Therefore,

$$G(\omega) = \frac{1}{2\pi}\int_\Gamma f(z)dz = \frac{1}{2\pi}\cdot \underset{\uparrow}{(-2\pi i)}\ \text{Res}(f; -2i) = -i\cdot\frac{e^{-i\omega(-2i)}}{-4i}$$

Clockwise orientation

$$= \frac{1}{4}e^{-2\omega}. \quad \checkmark$$

Since $F(t)$ is real-valued, a homework problem showed that the real part of $G(\omega)$ is an even function and the imaginary part of $G(\omega)$ is an odd function. In this case the imaginary part of $G(\omega)$ is zero, so it follows that $G(\omega) = \frac{1}{4}e^{-2|\omega|}$ for all $\omega \in \mathbf{R}$.

(b) State the Fourier inversion formula that expresses $F(t)$ in terms of $G(\omega)$, and verify by direct computation that it holds.

Inversion formula: $F(t) = \text{p.v.} \int_{-\infty}^{\infty} \frac{1}{4} e^{-2|\omega|} e^{i\omega t} \, d\omega$.

$$\int_{-\infty}^{\infty} \frac{1}{4} e^{-2|\omega|} e^{i\omega t} \, d\omega = \frac{1}{4} \int_{-\infty}^{0} e^{2\omega} e^{i\omega t} \, d\omega + \frac{1}{4} \int_{0}^{\infty} e^{-2\omega} e^{i\omega t} \, d\omega$$

$$= \frac{1}{4} \int_{-\infty}^{0} e^{(2+it)\omega} \, d\omega + \frac{1}{4} \int_{0}^{\infty} e^{(-2+it)\omega} \, d\omega$$

$$= \frac{1}{4} \cdot \frac{e^{(2+it)\omega}}{2+it} \Bigg|_{\omega=-\infty}^{0} + \frac{1}{4} \cdot \frac{e^{(-2+it)\omega}}{-2+it} \Bigg|_{\omega=0}^{\infty}$$

$$= \frac{1}{4} \cdot \frac{1}{2+it} (1-0) + \frac{1}{4} \cdot \frac{1}{-2+it} (0-1)$$

$$= \frac{1}{4} \left(\frac{1}{2+it} + \frac{1}{2-it} \right) = \frac{1}{4} \cdot \frac{4}{4+t^2} = \frac{1}{t^2+4}. \quad \checkmark$$

9. (10 points) Given a function $f(t)$, we denote its Laplace transform by $\mathcal{L}\{f\}(s)$ or simply by $\mathcal{L}\{f\}$.

Here is one of the entries in the textbook's table of Laplace transforms:*

$$\mathcal{L}\{\cos t\} = \mathrm{Re}\,\mathcal{L}\{e^{it}\} = \frac{s}{s^2+1}, \qquad \mathrm{Re}(s) > 0.$$

(a) This formula *must be wrong*, because $\mathrm{Re}\,\mathcal{L}\{e^{it}\}$ is always a real number, while $\frac{s}{s^2+1}$ is usually complex. In fact, it *is* true that $\mathcal{L}\{\cos t\} = \frac{s}{s^2+1}$ for all s with $\mathrm{Re}(s) > 0$, but the statement that $\mathcal{L}\{\cos t\} = \mathrm{Re}\,\mathcal{L}\{e^{it}\}$ is false. Find and explain the mistake in the following incorrect proof:

Let $\mathrm{Re}(s) > 0$. Then

$$\mathcal{L}\{\cos t\}(s) = \int_0^\infty \cos(t)e^{-st}dt \boxed{=} \int_0^\infty \mathrm{Re}\left(e^{it}e^{-st}\right)dt$$

$$= \mathrm{Re}\left(\int_0^\infty e^{it}e^{-st}dt\right) = \mathrm{Re}\,\mathcal{L}\{e^{it}\}(s).$$

This is false because if e^{-st} is not real,

$\cos(t)e^{-st} = \mathrm{Re}\left(e^{it}\right)e^{-st} \neq \mathrm{Re}\left(e^{it}e^{-st}\right)$.

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*If you have a very good memory, you may recall that the actual entry in the table is $\mathcal{L}\{\cos \omega t\} = \mathrm{Re}\,\mathcal{L}\{e^{i\omega t}\} = \frac{s}{s^2+\omega^2}$, for $\omega \in \mathbf{R}$ and $\mathrm{Re}(s) > 0$. The equation above is the special case $\omega = 1$.

(b) Prove that $\mathcal{L}\{\cos t\} = \dfrac{s}{s^2+1}$ for all s with $\mathrm{Re}(s) > 0$.

$$\mathcal{L}\{\cos t\}(s) = \int_0^\infty \cos(t)e^{-st}\,dt = \int_0^\infty \frac{e^{it}+e^{-it}}{2}\,e^{-st}\,dt$$

$$= \frac{1}{2}\int_0^\infty e^{(i-s)t}\,dt + \frac{1}{2}\int_0^\infty e^{(-i-s)t}\,dt$$

$$= \frac{1}{2}\cdot\frac{e^{(i-s)t}}{i-s}\bigg|_{t=0}^{\infty} + \frac{1}{2}\cdot\frac{e^{(-i-s)t}}{-i-s}\bigg|_{t=0}^{\infty}.$$

Since $\left|e^{(\pm i-s)t}\right| = e^{-\mathrm{Re}(s)\cdot t}$, if $\mathrm{Re}(s)>0$, the upper limits (at $t=\infty$) evaluate to 0. Therefore,

$$\mathcal{L}\{\cos t\}(s) = \frac{1}{2}\cdot\frac{1}{i-s}(0-1) + \frac{1}{2}\cdot\frac{1}{-i-s}(0-1)$$

$$= \frac{1}{2}\left(\frac{1}{s-i}+\frac{1}{s+i}\right) = \frac{1}{2}\cdot\frac{2s}{s^2+1} = \frac{s}{s^2+1}. \checkmark$$