Math 4220: Final Exam

December 11, 2015

Name:

Instructions: This exam is 150 minutes long. It has 14 pages including the cover and 9 questions worth a total of 100 points. No written or electronic aids are allowed.

Please fully explain all your answers. If you need more space to answer a question, use the back of the *preceding* sheet or the blank sheet at the end of the exam. Label your work clearly if you use extra space.

Academic integrity is expected of all Cornell University students at all times, whether in the presence or absence of members of the faculty. Understanding this, I declare I shall not give, use, or receive unauthorized aid in this examination.

Please sign below to indicate that you have read and agree to these instructions.

Signature:

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1. (10 points) (a) Let $f(z)$ be an analytic function such that $f(1+2i) = 4-i$ and $f'(1+2i) = 2+3i$. Write the linear approximation for $f(z)$ near $1+2i$.

(b) If $f(x+iy) = u(x,y) + iv(x,y)$, write the linear approximations for u and v near $(1,2)$. Use these to compute the partial derivatives $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, $\frac{\partial v}{\partial y}$ at $(1,2)$, and confirm that the Cauchy-Riemann equations hold.

2. (10 points) (a) What are the three values of $(-8)^{1/3}$? Express your answers in $re^{i\theta}$ form.

(b) Write an explicit formula for a function $f(z)$ which is a branch of $z^{1/3}$ where the branch cut is along the negative imaginary axis. You may use the notation $\arg_\theta(z)$ to denote the argument of z in the interval $(\theta, \theta + 2\pi]$. Compute $f(-8)$ and $f(8)$.

(c) Write an explicit formula for another function $g(z)$ which is also a branch of $z^{1/3}$ where the branch cut is along the negative imaginary axis, but for which $g(-8) \neq f(-8)$. Compute $g(-8)$ and $g(8)$.

3. (10 points) (a) Let γ_1 be the curve parametrized by $z(t) = e^{(1+i)t}$, $0 \le t \le 2\pi$. Let γ_2 be the curve parametrized by $z(t) = e^{(1-i)t}$, $0 \le t \le 2\pi$. Draw both curves γ_1 and γ_2 on the same graph, labeling the endpoints and the orientations.

(b) Compute $\displaystyle\int_{\gamma_1 - \gamma_2} \left(\frac{1}{z} + \frac{e^z}{2-z} \right) dz.$

4. (10 points) (a) Suppose $f(z)$ is analytic in a neighborhood of the origin, and $f(z) - \cos(z)$ has a zero of order 3 at the origin. What can be said about the values $f(0)$, $f'(0)$, $f''(0)$, $f'''(0)$?

(b) Let $g(z) = \dfrac{1}{f'(z) + z}$. Does g have a singularity at the origin? If so, classify it as a removable singularity, a pole whose order you specify, or an essential singularity. Justify your answer.

5. (10 points) Suppose $f(z)$ is analytic on the annulus $A = \{r < |z| < R\}$. Prove that there exist functions $g(z)$ and $h(z)$ such that g is analytic on the disk $\{|z| < R\}$, h is analytic on the region $\{|z| > r\}$, and $f(z) = g(z) + h(z)$ for all z in A.

6. (12 points) For fixed $a \in \mathbf{R}$, compute p.v. $\displaystyle\int_{-\infty}^{\infty} \frac{\cos(t)}{t-a}\, dt$.

Hint: Use $\cos(t) = \mathrm{Re}(e^{it})$.

7. (10 points) (a) Find constants c_n such that $\sin(t) + \cos(3t) = \sum_{n=-\infty}^{\infty} c_n e^{int}$.

(b) Suppose c_n is given. Find a solution of the form $f(t) = A_n e^{int}$ to the differential equation $f'(t) + 5f(t) = c_n e^{int}$.

(c) Use parts (a) and (b) to find a periodic solution to the differential equation $f'(t) + 5f(t) = \sin(t) + \cos(3t)$. You do not need to simplify your answer.

8. (18 points) (a) Let $F(t) = \dfrac{1}{t^2+4}$ and let $G(\omega)$ be its Fourier transform. By evaluating the formula for G as a principal value integral, show that $G(\omega) = \frac{1}{4}e^{-2\omega}$ when $\omega \geq 0$.

Since $F(t)$ is real-valued, a homework problem showed that the real part of $G(\omega)$ is an even function and the imaginary part of $G(\omega)$ is an odd function. In this case the imaginary part of $G(\omega)$ is zero, so it follows that $G(\omega) = \frac{1}{4}e^{-2|\omega|}$ for all $\omega \in \mathbf{R}$.

(b) State the Fourier inversion formula that expresses $F(t)$ in terms of $G(\omega)$, and verify by direct computation that it holds.

9. (10 points) Given a function $f(t)$, we denote its Laplace transform by $\mathcal{L}\{f\}(s)$ or simply by $\mathcal{L}\{f\}$.

Here is one of the entries in the textbook's table of Laplace transforms:*

$$\mathcal{L}\{\cos t\} = \operatorname{Re}\mathcal{L}\{e^{it}\} = \frac{s}{s^2+1}, \qquad \operatorname{Re}(s) > 0.$$

(a) This formula *must be wrong*, because $\operatorname{Re}\mathcal{L}\{e^{it}\}$ is always a real number, while $\frac{s}{s^2+1}$ is usually complex. In fact, it *is* true that $\mathcal{L}\{\cos t\} = \frac{s}{s^2+1}$ for all s with $\operatorname{Re}(s) > 0$, but the statement that $\mathcal{L}\{\cos t\} = \operatorname{Re}\mathcal{L}\{e^{it}\}$ is false. Find and explain the mistake in the following incorrect proof:

Let $\operatorname{Re}(s) > 0$. Then

$$\begin{aligned}\mathcal{L}\{\cos t\}(s) = \int_0^\infty \cos(t)e^{-st}dt &= \int_0^\infty \operatorname{Re}\left(e^{it}e^{-st}\right)dt \\ &= \operatorname{Re}\left(\int_0^\infty e^{it}e^{-st}dt\right) = \operatorname{Re}\mathcal{L}\{e^{it}\}(s).\end{aligned}$$

*If you have a very good memory, you may recall that the actual entry in the table is $\mathcal{L}\{\cos \omega t\} = \operatorname{Re}\mathcal{L}\{e^{i\omega t}\} = \frac{s}{s^2+\omega^2}$, for $\omega \in \mathbf{R}$ and $\operatorname{Re}(s) > 0$. The equation above is the special case $\omega = 1$.

(b) Prove that $\mathcal{L}\{\cos t\} = \dfrac{s}{s^2+1}$ for all s with $\mathrm{Re}(s) > 0$.

(This page for scratch work.)