MATH 4220 HOMEWORK 1 SOLUTION SET 9/7/2015

§1.1 #8: $\dfrac{(8+2i)-(1-i)}{(2+i)^2} = \dfrac{7+3i}{(4-1)+4i} = \dfrac{(7+3i)}{(3+4i)}\cdot\dfrac{(3-4i)}{(3-4i)}$

$$= \frac{(21+12)+(9i-28i)}{9+16} = \boxed{\frac{33}{25} - \frac{19}{25}i}$$

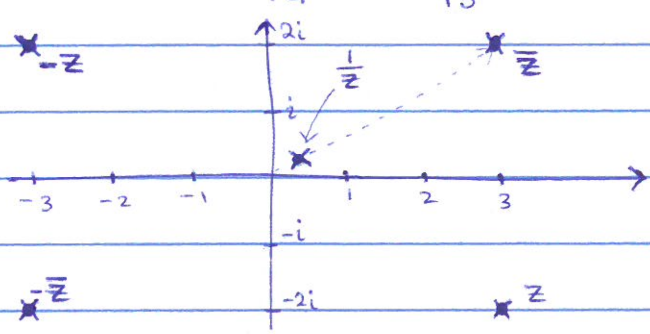
#14: Show $\mathrm{Re}(iz) = -\mathrm{Im}(z)$ for every complex number z.

Say $z = x+iy$ where x, y are real.

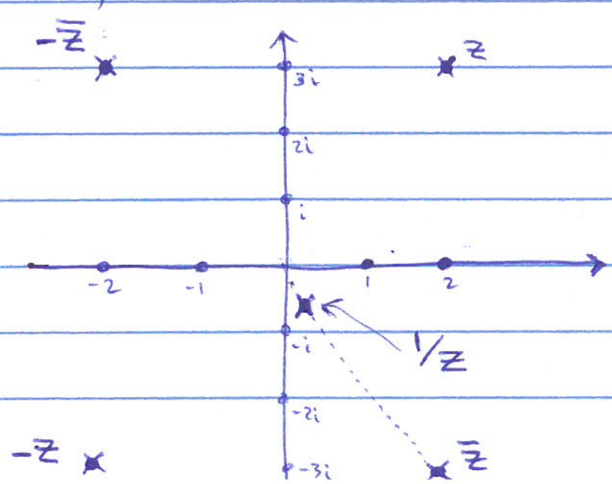
then $iz = ix + iiy = -y + ix$

So $\mathrm{Re}(iz) = \mathrm{Re}(-y+ix) = -y = -\mathrm{Im}(z)$ ∎

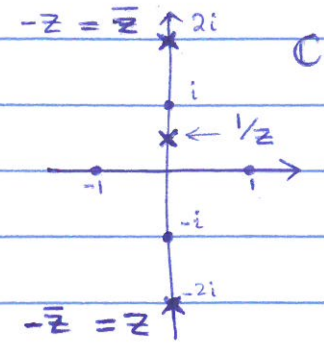
§1.2 #4: $z = 3-2i$, $\dfrac{1}{z} = \dfrac{\bar{z}}{|z|^2} = \dfrac{3+2i}{13}$, so we see:



$z = 2+3i$, $1/z = \bar{z}/|z|^2 = \dfrac{2-3i}{13}$



$z = -2i, \quad \frac{1}{z} = \frac{1}{-2i} = +\frac{i}{2}$



#10: Prove $|Re(z)| \leq |z|$ and $|Im(z)| \leq |z|$.

Say $z = x + iy$ where x, y are real.

We know $y^2 \geq 0$

so, $x^2 + y^2 \geq x^2$ adding x^2 to both sides,

and, $\sqrt{x^2+y^2} \geq \sqrt{x^2} \geq 0$ since both sides are non-negative,

thus $|z| \geq |x| = |Re(z)|$

Similarly $x^2 \geq 0$ gives $x^2 + y^2 \geq y^2$

and taking square roots we get: $|z| \geq |y| = |Im(z)|$

$\blacksquare$

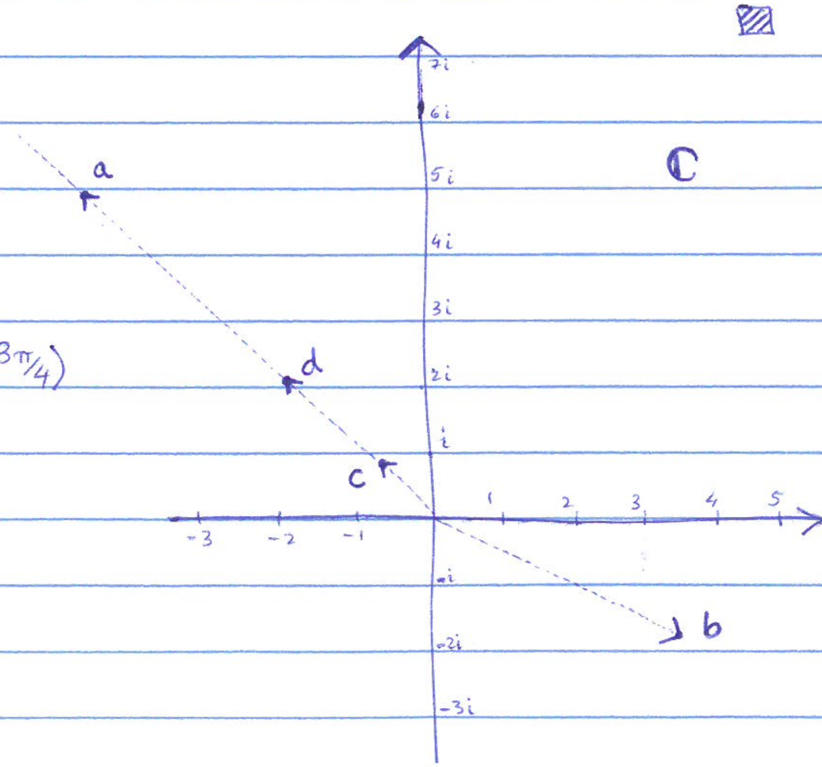
§1.3 #6: a) $7 \operatorname{cis}(3\pi/4) = 7e^{i3\pi/4}$

b) $4 \operatorname{cis}(-\pi/6)$

c) $\operatorname{cis}(3\pi/4)$

d) $3 \operatorname{cis}(27\pi/4) = 3 \operatorname{cis}(6\pi + 3\pi/4)$

$= 3 \operatorname{cis}(3\pi/4)$

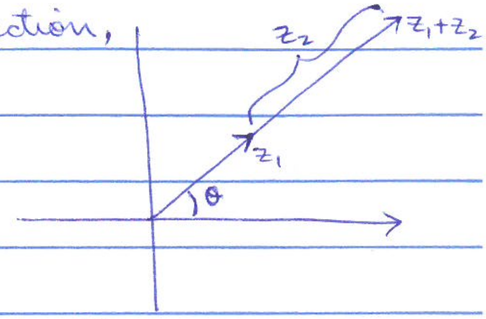


#8: **Show** geometrically (for nonzero z_1, z_2) $|z_1+z_2| = |z_1|+|z_2|$ if and only if z_1, z_2 have the same argument:

First the "if" part: Say z_1, z_2 have the same argument (say θ)

$z_1 = r_1 e^{i\theta}$, $z_2 = r_2 e^{i\theta}$.

Then z_1+z_2 also points in the same direction, (by definition of vector addition), & has length $|z_1|+|z_2|$.

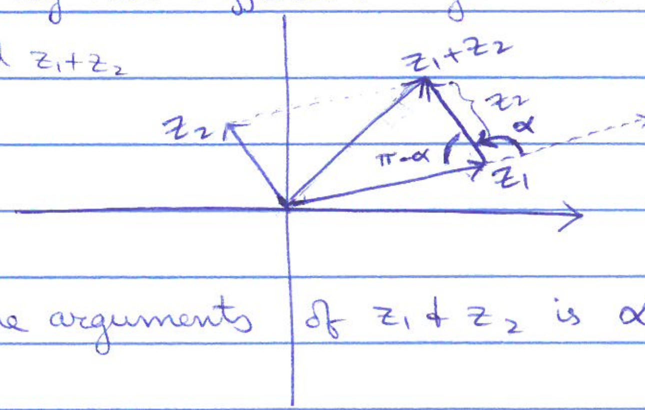


[You could also directly check

$$|z_1+z_2| = |r_1e^{i\theta}+r_2e^{i\theta}| = |r_1+r_2||e^{i\theta}|$$
$$= r_1+r_2 = |z_1|+|z_2|$$]

Now, the "only if" part: Suppose $|z_1+z_2| = |z_1|+|z_2|$ and we want to show that z_1, z_2 have the same argument.

For contradiction, assume they had different arguments. Then z_1 followed by z_2 and z_1+z_2 would form a triangle with positive area.



Say the difference between the arguments of z_1 & z_2 is α.

Using cosine rule we get:

$$|z_1+z_2|^2 = |z_1|^2+|z_2|^2 - 2|z_1||z_2|\cos(\pi-\alpha)$$

$$\Rightarrow (|z_1|+|z_2|)^2 = |z_1|^2+|z_2|^2 - 2|z_1|\cdot|z_2|\cos(\pi-\alpha) \quad \left(\text{using } |z_1|+|z_2| = |z_1+z_2|\right)$$

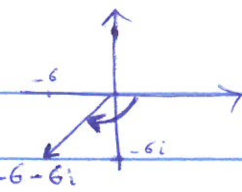
$$\Rightarrow 2|z_1|\cdot|z_2| = 2|z_1||z_2|\cos(\alpha), \quad \left(\begin{array}{l}\text{using } \cos(\pi-\alpha) = -\cos\alpha \\ \text{and expanding the square}\end{array}\right)$$

$$\Rightarrow \cos(\alpha) = 1$$

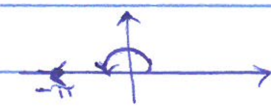
So $\alpha = 0$, is the difference in angle between z_1 & z_2. Thus they have the same argument.

$\blacksquare$

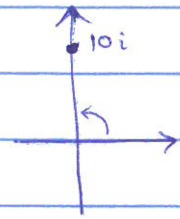
#12: a) $Arg(-6-6i) = -3\pi/4$



b) $Arg(-\pi) = +\pi$

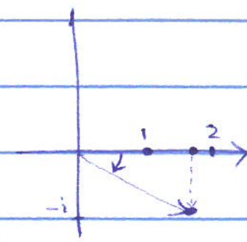


c) $Arg(10i) = \pi/2$



d) $Arg(\sqrt{3}-i) = \tan^{-1}(1/\sqrt{3}) = \sin^{-1}(1/2)$

$= -\pi/6$



[Remember Arg (with a capital "A") is between $(-\pi, \pi]$]

§1.4 #2: a) $\dfrac{e^{3i}-e^{-3i}}{2i} = Im(e^{3i}) = \boxed{\sin(3)}$

b) $2e^{3+i\pi/6} = 2e^3 e^{i\pi/6} = 2e^3(\cos\pi/6 + i\sin\pi/6)$

$= 2e^3\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right) = e^3(\sqrt{3}+i) = \boxed{\sqrt{3}e^3 + e^3 i}$

c) $z = 4e^{i\pi/3} = 4(\cos\pi/3 + i\sin\pi/3) = 4(1/2 + i\sqrt{3}/2) = 2 + i2\sqrt{3}$

So $e^z = e^{2+i2\sqrt{3}} = e^2(\cos(2\sqrt{3}) + i\sin(2\sqrt{3})) = \boxed{e^2\cos(2\sqrt{3}) + i e^2 \sin(2\sqrt{3})}$

#10: • $Re(z) \le 0$, so $|e^z| = |e^{Re(z)+i\,Im(z)}|$

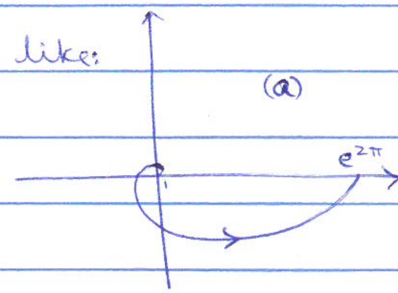
$= \underbrace{|e^{Re(z)}|}_{\text{always positive}} \cdot \underbrace{|e^{i\,Im(z)}|}_{\text{always } 1} = e^{Re(z)}$

Thus $|e^z|$ is always $e^{Re(z)}$, so given $Re(z)$ is negative we get $|e^z| = e^{Re(z)} = \dfrac{1}{e^{|Re(z)|}} \leq 1$

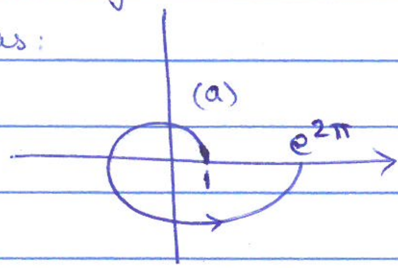
#18: Sketch for $0 \le t \le 2\pi$:

a) $z(t) = e^{(1+i)t} = e^{t}e^{it}$

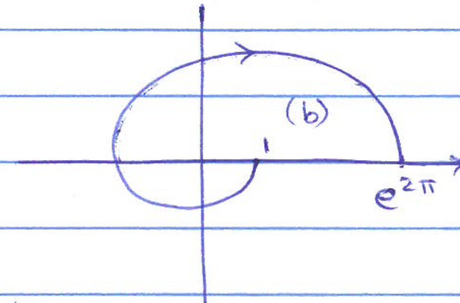
The actual plot will look something like:



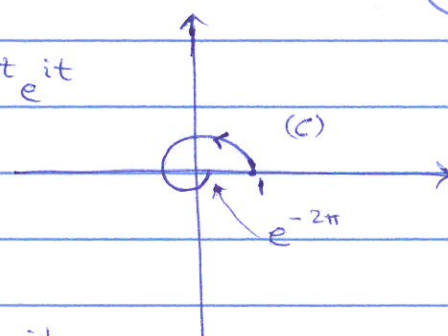
But schematically we can depict it as:



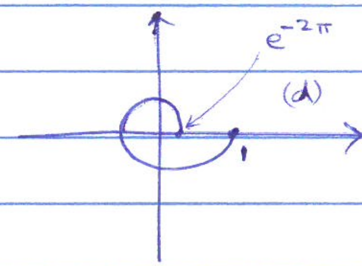
b) $z(t) = e^{(1-i)t} = \underbrace{e^{t}}_{\text{length}}\underbrace{e^{-it}}_{\text{angle}}$



c) $z(t) = e^{(-1+i)t} = e^{-t}e^{it}$



d) $z(t) = e^{(-1-i)t} = e^{-t}e^{-it}$



§1.5 #4: a) $(\sqrt{3}-i)^7 = (2e^{-i\pi/6})^7 = 2^7(\cos(-\pi/6) + i\sin(-\pi/6))^7$

$= e^{-i7\pi/6} \quad = 2^7(\cos(-7\pi/6) + i\sin(-7\pi/6))$

$= 128(-\cos(\pi/6) + i\sin(\pi/6))$

$= 128(-\sqrt{3}/2 + i\, 1/2) = -64\sqrt{3} + i\,64.$

b) $(1+i)^{95} = (\sqrt{2}\, e^{i\pi/4})^{95} = 2^{47}\sqrt{2}\, e^{i95\pi/4} = 2^{47}\sqrt{2}\, e^{i(24\pi - \pi/4)}$

$= 2^{47}\sqrt{2}\,\frac{(1-i)}{\sqrt{2}} = 2^{47}(1-i)$.

#10: $z^4 + 1 = 0$ So $z^4 = -1 = 1 \cdot e^{i\pi}$

If $z = re^{i\theta}$, $z^4 = r^4 e^{4i\theta} = 1e^{i\pi} = e^{-i\pi} = e^{3i\pi} = e^{-3i\pi}$, etc.

So $4\theta = \pi, -\pi, 3\pi$ or -3π (for instance).

So the four roots are $e^{\pm i\pi/4}$, $e^{\pm i3\pi/4}$, and they come in conjugate pairs. We thus have:

$$z^4 + 1 = \left(z - e^{i\frac{\pi}{4}}\right)\left(z - e^{-i\frac{\pi}{4}}\right)\left(z - e^{3i\frac{\pi}{4}}\right)\left(z - e^{-3i\frac{\pi}{4}}\right)$$

$$= \left(z^2 - z\left(e^{i\frac{\pi}{4}} + e^{-i\frac{\pi}{4}}\right) + 1\right)\left(z^2 - z\left(e^{3i\frac{\pi}{4}} + e^{-3i\frac{\pi}{4}}\right) + 1\right)$$

$$= \left(z^2 - z\left(2\,\mathrm{Re}\left(e^{i\frac{\pi}{4}}\right)\right) + 1\right)\left(z^2 - z\left(2\,\mathrm{Re}\left(e^{i3\frac{\pi}{4}}\right)\right) + 1\right)$$

$$= \left(z^2 - \sqrt{2}\,z + 1\right)\left(z^2 + \sqrt{2}\,z + 1\right)$$

Q.E.D.