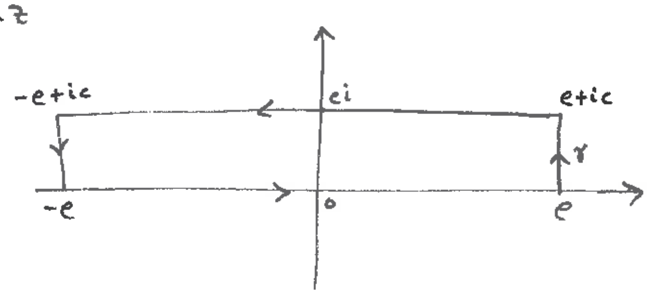
EXTRA: $c \in \mathbb{R}$, $\gamma_c : z(t) = t + ic$, $t \in \mathbb{R}$.

Define $\displaystyle\int_{-\infty+ci}^{\infty+ci} f(z)\,dz = \int_{\gamma_c} f(z)\,dz$



(a) Let γ be a vertical segment from ρ to $\rho+ic$

Then
$$\left|\int_\gamma e^{-z^2}dz\right| \le \int_\gamma |e^{-z^2}|\,dz$$
$$\le c \cdot \max_\gamma |e^{-z^2}|$$
$$= c \cdot \max_\gamma |e^{y^2-x^2-2xyi}| \quad \text{where } z = x+iy.$$
$$= c \cdot \max_\gamma (e^{y^2-x^2})$$
$$= c \cdot e^{-\rho^2+c^2}$$

So $\displaystyle\left|\int_\gamma e^{-z^2}dz\right| \le c \cdot e^{c^2}e^{-\rho^2}$ and $\to 0$ as $\rho \to \infty$.

Similarly if γ' is the vertical segment from $-\rho+ic$ to $-\rho$ then
$$\left|\int_{\gamma'} e^{-z^2}dz\right| \le c \cdot \max_{\gamma'} (e^{y^2-x^2}) = c \cdot e^{c^2}e^{-\rho^2} \to 0 \text{ as } \rho \to \infty.$$

(b) Using the contour above and letting $\rho \to \infty$ we get that
$$\int_{-\infty}^{\infty} e^{-z^2}dz - \int_{-\infty+ic}^{\infty+ic} e^{-z^2}dz = 0 \qquad \text{since } e^{-z^2} \text{ has no poles.}$$

Thus $\displaystyle\int_{-\infty+ic}^{\infty+ic} e^{-z^2}dz = \int_{-\infty}^{\infty} e^{-z^2}dz = \sqrt{\pi}$

8.2.1 a) $F(t) = e^{-|t|}$

$$G(w) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-|t|} e^{-iwt}\, dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{0} e^{t(1-iw)}\, dt + \frac{1}{2\pi}\int_{0}^{\infty} e^{-t(1+iw)}\, dt$$

$$= \frac{1}{2\pi}\left.\frac{e^{t(1-iw)}}{1-iw}\right|_{-\infty}^{0} + \frac{1}{2\pi}\left.\frac{e^{-t(1+iw)}}{-(1+iw)}\right|_{0}^{\infty} \qquad (w \in \mathbb{R}).$$

$$= \frac{1}{2\pi}\left(\frac{1}{1-iw} - 0\right) + \frac{1}{2\pi}\left(0 - \frac{1}{-(1+iw)}\right)$$

$$= \frac{1}{2\pi}\,\frac{2}{1+w^2} = \boxed{\frac{1}{\pi(1+w^2)}}$$

The inversion formula then is:

$$F(t) = e^{-|t|} = \text{p.v.}\int_{-\infty}^{\infty} \frac{e^{iwt}}{\pi(1+w^2)}\, dt \qquad \text{(as } F \text{ is continuous)}$$

b) $F(t) = e^{-t^2}$

$$G(w) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-t^2 - iwt}\, dt$$

Now $-(t^2 + iwt) = -\left(t^2 + 2t\left(\frac{iw}{2}\right) + \left(\frac{iw}{2}\right)^2 - \left(\frac{iw}{2}\right)^2\right) = -\left(\left(t + i\frac{w}{2}\right)^2 + \frac{w^2}{4}\right)$

So $$G(w) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-(t+iw/2)^2} e^{-w^2/4}\, dt$$

$$= \frac{1}{2\pi} e^{-w^2/4} \int_{-\infty + iw/2}^{\infty + iw/2} e^{-z^2}\, dz = \boxed{\frac{e^{-w^2/4}}{2\pi}\sqrt{\pi}}$$

And $$F(t) = e^{-t^2} = \text{p.v.}\int_{-\infty}^{\infty} \frac{e^{-w^2/4} e^{iwt}}{2\sqrt{\pi}}\, dw$$

d) $F(t) = \dfrac{\sin(t)}{t}$

$$G(w) = \text{p.v.} \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\sin(t)}{t} e^{-iwt}\, dt$$

Let $t = s\pi \qquad dt = \pi\, ds$

$$G(w) = \text{p.v.} \frac{1}{2\pi}\, \pi \int_{-\infty}^{\infty} \frac{\sin(s\pi)}{s\pi} e^{is(-w\pi)}\, ds$$

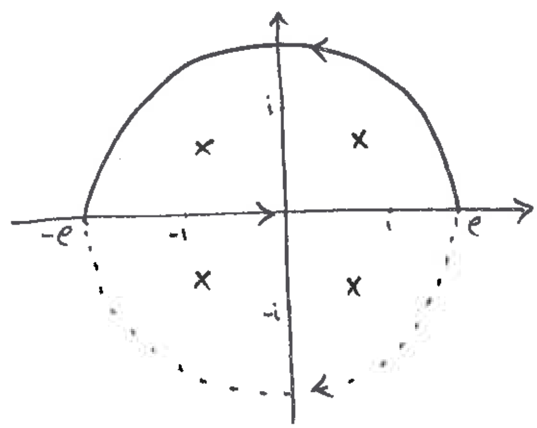
So
$$G(w) = \boxed{\begin{cases} 1/2, & |w\pi| < \pi \quad \text{or} \quad |w| < 1 \\ 0, & |w\pi| > \pi \qquad\qquad |w| > 1 \\ 1/4, & |w\pi| = \pi \qquad\qquad w = \pm 1 \end{cases}}$$

Then
$$F(t) = \frac{\sin(t)}{t} = \text{p.v.} \int_{-\infty}^{\infty} G(w) e^{iwt}\, dw$$

$$= \int_{-1}^{1} \frac{e^{iwt}}{2}\, dw$$

8.2.3 a) $F(t) = \dfrac{1}{t^4 + 1}$

$$G(w) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-iwt}}{t^4+1}\, dt$$



The integrand has four simple poles at $e^{\pm i\pi/4}$ and $e^{\pm i3\pi/4}$

when $w > 0$, we the lower contour and Jordan's lemma gives us

$$G(w) = -i\left(\text{Res}\left(e^{-i\pi/4}\right) + \text{Res}\left(e^{-3\pi i/4}\right)\right)$$

$$= \frac{-i\, e^{-iw\left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)}}{-i\sqrt{2}\cdot\sqrt{2}\,(\sqrt{2} - \sqrt{2}i)} + \frac{-i\, e^{-iw\left(-\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)}}{-i\sqrt{2}\,(-\sqrt{2})(-\sqrt{2} - \sqrt{2}i)}$$

$$= \frac{e^{-w/\sqrt{2}}}{2\sqrt{2}} \left(\frac{e^{-iw/\sqrt{2}}}{1-i} + \frac{e^{iw/\sqrt{2}}}{1+i} \right)$$

Similarly, when $w<0$, use the upper contour to get

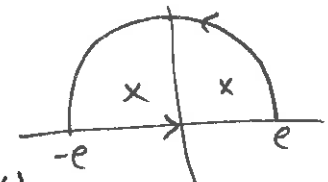
$$G(w) = i\,\mathrm{Res}(e^{i\pi/4}) + i\,\mathrm{Res}(e^{3\pi i/4})$$

$$= \frac{i\, e^{-iw\left(\frac{1}{\sqrt2}+\frac{i}{\sqrt2}\right)}}{i\sqrt2\cdot\sqrt2\cdot(\sqrt2+i\sqrt2)} + \frac{i\cdot e^{-iw\left(-\frac{1}{\sqrt2}+\frac{i}{\sqrt2}\right)}}{i\sqrt2(-\sqrt2)(-\sqrt2+i\sqrt2)}$$

$$= \frac{e^{w/\sqrt2}}{2\sqrt2}\left(\frac{e^{-iw/\sqrt2}}{1+i} + \frac{e^{iw/\sqrt2}}{1-i}\right)$$

Finally, for $w=0$ (although unnecessary for the inversion formula)

$$G(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{1+t^4}\,dt$$



$$= i\,\mathrm{Res}\left(\frac{1}{1+t^4}, e^{i\pi/4}\right) + i\,\mathrm{Res}\left(\frac{1}{1+t^4}, e^{i3\pi/4}\right)$$

$$= i\left(\frac{1}{i2\sqrt2(1+i)} + \frac{1}{i2\sqrt2(1-i)}\right) = \frac{1}{2\sqrt2}\left(\frac{1}{1+i}+\frac{1}{1-i}\right) = \frac{1}{2\sqrt2}$$

Thus for $w\in\mathbb{R}$ $\quad G(w) = \boxed{\frac{e^{-|w|/\sqrt2}}{2\sqrt2}\left(\frac{e^{-i|w|/\sqrt2}}{1-i} + \frac{e^{i|w|/\sqrt2}}{1+i}\right)}$

The inversion formula becomes:

$$\frac{1}{t^4+1} = \int_{-\infty}^{0}\frac{e^{w/\sqrt2}}{2\sqrt2}\left(\frac{e^{-iw/\sqrt2}}{1+i} + \frac{e^{iw/\sqrt2}}{1-i}\right)e^{iwt}\,dw$$

$$+ \int_{0}^{\infty}\frac{e^{-w/\sqrt2}}{2\sqrt2}\left(\frac{e^{-iw/\sqrt2}}{1-i} + \frac{e^{iw/\sqrt2}}{1+i}\right)e^{iwt}\,dw$$

So let us verify this:

$$\text{R.H.S.} = \int_{-\infty}^{0} \frac{e^{w\left(\frac{1}{\sqrt2}-\frac{i}{\sqrt2}+it\right)}}{2\sqrt2(1+i)} + \frac{e^{w\left(\frac{1}{\sqrt2}+\frac{i}{\sqrt2}+it\right)}}{2\sqrt2\,(1-i)}\,dw$$

$$+\int_{0}^{\infty} \frac{e^{w\left(-\frac{1}{\sqrt2}-i/\sqrt2+it\right)}}{2\sqrt2\,(1-i)} + \frac{e^{w\left(-\frac{1}{\sqrt2}+i/\sqrt2+it\right)}}{2\sqrt2\,(1+i)}\,dw \,.$$

$$= \frac{1}{2\sqrt2(1+i)\left(\frac{1}{\sqrt2}-\frac{i}{\sqrt2}+it\right)} - 0 + \frac{1}{2\sqrt2(1-i)\left(\frac{1}{\sqrt2}+\frac{i}{\sqrt2}+it\right)} - 0$$

$$+\,0 - \frac{1}{2\sqrt2(1-i)\left(\frac{-1}{\sqrt2}-\frac{i}{\sqrt2}+it\right)} + 0 - \frac{1}{2\sqrt2(1+i)\left(\frac{-1}{\sqrt2}+\frac{i}{\sqrt2}+it\right)} \,.$$

$$= \frac{1}{2}\left(\frac{1}{1+i}\left(\frac{1}{1-i+it\sqrt2} - \frac{1}{-1+i+it\sqrt2}\right) + \frac{1}{1-i}\left(\frac{1}{1+i+it\sqrt2} - \frac{1}{-1-i+it\sqrt2}\right)\right)$$

$$= \frac{-(1-i)}{(1+i)[2i-2t^2]} + \frac{-(1+i)}{(1-i)(-2i-2t^2)} \qquad \left(\text{Note } \frac{1+i}{1-i} = i\right)$$

$$= \frac{1}{2}\left(\frac{+i}{i-t^2} + \frac{-i}{-i-t^2}\right)$$

$$= \frac{i}{2}\left(\frac{+1}{i-t^2} + \frac{1}{i+t^2}\right) = \frac{i}{2}\left(\frac{2i}{(i-t^2)(i+t^2)}\right) = \frac{-1}{-1-t^4} = \frac{1}{1+t^4}$$

c) $F(t) = e^{-t^2}$, $G(w) = \dfrac{e^{-w^2/4}}{2\sqrt\pi}$ (8.2.1([illegible]))

And $$\text{p.v.}\int_{-\infty}^{\infty} \frac{e^{-w^2/4}\,e^{iwt}}{2\sqrt\pi}\,dw = \text{p.v.}\int_{-\infty}^{\infty} \frac{e^{-\left(\frac{w}{2}-it\right)^2-t^2}}{2\sqrt\pi}\,dw$$

as $-\dfrac{w^2}{4} + iwt = -\left(\dfrac{w}{2}-it\right)^2 - t^2$

So $$\text{p.v.} \int_{-\infty}^{\infty} \frac{e^{-\omega^2/4} e^{i\omega t}}{2\sqrt{\pi}} \, d\omega$$

$$= \frac{e^{-t^2}}{2\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\left(\frac{\omega}{2} - it\right)^2} d\omega$$

$$= \frac{e^{-t^2}}{2\sqrt{\pi}} \int_{-\infty - it}^{\infty - it} e^{-z^2} \, 2\,dz \qquad \text{set } \frac{\omega}{2} - it = z$$

$$= \frac{e^{-t^2}}{2\sqrt{\pi}} (2\sqrt{\pi}) = e^{-t^2} = F(t)$$

$\boxtimes$

8.2.4: $F(t)$ a real function.

$$G(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(t) e^{-i\omega t} \, dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(t) \cos(\omega t) \, dt - \frac{1}{2\pi} \int_{-\infty}^{\infty} F(t) \sin(\omega t) \, dt$$

So $$\text{Re}(G(\omega)) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(t) \cos(\omega t) \, dt = f(\omega) \quad \text{say}$$

and $$\text{Im}(G(\omega)) = \frac{-1}{2\pi} \int_{-\infty}^{\infty} F(t) \sin(\omega t) \, dt = g(\omega) \quad \text{say.}$$

Then $$f(-\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(t) \cos(-\omega t) \, dt = f(\omega) \quad \text{So } f \text{ is even.}$$

$$g(-\omega) = \frac{-1}{2\pi} \int_{-\infty}^{\infty} F(t) \sin(-\omega t) \, dt = -g(+\omega) \quad \text{So } g \text{ is odd.}$$

$\boxtimes$

8.2.6 a) $f'' + f' + f = e^{-t^2}$

Say $G(w)$ is the Fourier transform of f.

And we know the Fourier transform of e^{-t^2} is $\dfrac{e^{-w^2/4}}{2\sqrt{\pi}}$

If $f(t) = \text{p.v.} \int_{-\infty}^{\infty} G(w)\, e^{iwt}\, dw$

then $f'(t) = \text{p.v.} \int_{-\infty}^{\infty} iw\, G(w)\, e^{iwt}\, dw$

and $f''(t) = \text{p.v.} \int_{-\infty}^{\infty} -w^2 G(w)\, e^{iwt}\, dw$

So $\text{p.v.}\int_{-\infty}^{\infty} e^{iwt} \dfrac{e^{-w^2/4}}{2\sqrt{\pi}}\, dw = e^{-t^2} = f'' + f' + f = \text{p.v.} \int_{-\infty}^{\infty} G(w)(1 + iw - w^2)\, e^{iwt}\, dw$

And $G(w) = \dfrac{1}{2\sqrt{\pi}} \dfrac{e^{-w^2/4}}{1 + iw - w^2}$

So $f(t) = \text{p.v.}\int_{-\infty}^{\infty} \dfrac{1}{2\sqrt{\pi}} \dfrac{e^{-w^2/4}}{1 + iw - w^2}\, e^{iwt}\, dw$ is a solution to the equation.

c) $f'' + 2f' + 3f = \begin{cases} 1 & |t| < 1 \\ 0 & \text{otherwise} \end{cases} = g(t)$ say.

Fourier transform of g is given by $\dfrac{1}{2\pi}\int_{-\infty}^{\infty} g(t)\, e^{-iwt}\, dt =$

$= \dfrac{1}{2\pi}\int_{-1}^{1} e^{-iwt}\, dt = \dfrac{1}{2\pi}\left(\dfrac{e^{-iw}}{-iw} - \dfrac{e^{iw}}{-iw}\right) = \dfrac{2i\sin w}{+2\pi i w} = +\dfrac{\sin(w)}{\pi w}$

If the Fourier transform of f is G then we have:

$$f(t) = \text{p.v.}\int_{-\infty}^{\infty} G(w)\, e^{iwt}\, dw$$

So $$f''(t) + 2f'(t) + 3f(t) = \text{p.v.}\int_{-\infty}^{\infty} G(w)\,(3 + 2iw - w^2)\, e^{iwt}\, dw$$

$$= \text{p.v.}\int_{-\infty}^{\infty} \frac{\sin(w)}{\pi w}\, e^{iwt}\, dw \quad \left(= \begin{cases} \tfrac{1}{2} & |t| = 1 \\ 1 & |t| < 1 \\ 0 & \text{otherwise} \end{cases} \right)$$

So set $$G(w) = \frac{\sin(w)}{\pi w (3 + 2iw - w^2)}$$

Thus $$f(t) = \text{p.v.}\int_{-\infty}^{\infty} \frac{\sin(w)\, e^{iwt}}{\pi w (3 + 2iw - w^2)}\, dw$$ is a solution.

8.2.7: $G(w)$ is the F.T of ~~[illegible]~~ $f(x)$.

If $$T(x,t) = \int_{-\infty}^{\infty} G(w)\, e^{iwx} e^{-w^2 t}\, dw,$$ then — converges when $t > 0$.

$$\frac{\partial T}{\partial t}(x,t) = \int_{-\infty}^{\infty} -w^2 G(w)\, e^{iwx} e^{-w^2 t}\, dw$$ whereas,

$$\frac{\partial^2 T}{\partial x^2}(x,t) = \int_{-\infty}^{\infty} (iw)^2 G(w)\, e^{iwx} e^{-w^2 t}\, dw = \frac{\partial T}{\partial t}(x,t).$$

Moreover, $$T(x,0) = \int_{-\infty}^{\infty} G(w)\, e^{iwx}\, dw = f(x).$$

Thus T solves the equations: $T_t = T_{xx}$ and $T(x,0) = f(x)$.

Finally, using the formula for $G(w)$, we get

$$T(x,t) = \int_{-\infty}^{\infty} \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} f(\xi)\, e^{-iw\xi}\, d\xi \right) e^{iwx} e^{-w^2 t}\, dw$$

$$\Rightarrow \quad T(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi)\, e^{-iw\xi}\, e^{iwx}\, e^{-w^2 t}\, dw\, d\xi$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\xi)\left(\int_{-\infty}^{\infty} e^{-iw\xi + iwx - w^2 t}\, dw\right) d\xi$$

Note $\quad -(w^2 t + iwx - iw\xi) = -\left((w\sqrt{t})^2 + 2w\sqrt{t}\left(\frac{i(x-\xi)}{2\sqrt{t}}\right)\right)$

$$= -\left(w\sqrt{t} + \frac{i(x-\xi)}{2\sqrt{t}}\right)^2 - \frac{(x-\xi)^2}{4t}$$

Thus, $\quad T(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\xi)\left(\int_{-\infty}^{\infty} e^{-\left(w\sqrt{t} + \frac{i(x-\xi)}{2\sqrt{t}}\right)^2} e^{-\frac{(x-\xi)^2}{4t}}\, dw\right) d\xi.$

Set $s = w\sqrt{t} + \frac{i(x-\xi)}{\sqrt{t}}$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{f(\xi)}{\sqrt{t}}\left(\int_{-\infty + \frac{i(x-\xi)}{2\sqrt{t}}}^{\infty + \frac{i(x-\xi)}{2\sqrt{t}}} e^{-s^2}\, ds\right) e^{-\frac{(x-\xi)^2}{4t}}\, d\xi.$$

$$= \frac{\sqrt{\pi}}{2\pi\sqrt{t}}\int_{-\infty}^{\infty} f(\xi)\, e^{-(x-\xi)^2/4t}\, d\xi \qquad \boxed{/\!/}.$$

8.2.8:

$$u(x,t) = \int_{-\infty}^{\infty} G(w)\, e^{iwx}\cos(wt)\, dw$$

$$\frac{\partial^2 u}{\partial x^2}(x,t) = \int_{-\infty}^{\infty} G(w)\,(iw)^2\, e^{iwx}\cos(wt)\, dw = \frac{\partial^2 u}{\partial t^2}(x,t).$$

and $\quad u(x,0) = \int_{-\infty}^{\infty} G(w)\, e^{iwx}\, dw = f(x). \qquad$ (given).

whereas $\quad \frac{\partial u}{\partial t}(x,0) = \int_{-\infty}^{\infty} G(w)(-w)\, e^{iwx}\sin(0)\, dw = 0.$

If instead $u(x,t) = \int_{-\infty}^{\infty} G(w)\, e^{iwx} \frac{\sin(wt)}{w}\, dw$

then $u_{tt} = u_{xx}$ still but

$u(x,0) = 0$ and

$$u_t(x,0) = \int_{-\infty}^{\infty} \frac{G(w)\, e^{iwx}}{w}\, w \cos(0)\, dw = f(x).$$

Thus for the general condition $u(x,0) = f_1(x)$ and

$$\frac{\partial u}{\partial t}(x,0) = f_2(x)$$

with $G_1(w)$ the F.T. of $f_1(x)$ and

$G_2(w)$ the F.T. of $f_2(x)$ we set

$$u(x,t) = \int_{-\infty}^{\infty} \left(G_1(w)\, e^{iwx} \cos(wt) + G_2(w)\, e^{iwx} \frac{\sin(wt)}{w} \right) dw$$