

## Math 4220: Homework 10

Due Thursday, December 3 in class.

Extra problem: Fix  $c \in \mathbf{R}$  and let  $\gamma_c$  be the infinite-length horizontal contour parametrized by  $z(t) = t + ic$ ,  $t \in \mathbf{R}$ . We invent the notation

$$\int_{-\infty+ic}^{\infty+ic} f(z)dz \quad \text{to mean} \quad \int_{\gamma_c} f(z)dz.$$

The purpose of this problem is to evaluate this integral when  $f(z) = e^{-z^2}$ . It can be shown without using complex analysis that

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}. \quad (1)$$

In what follows, you will use formula (1) to compute

$$\int_{-\infty+ic}^{\infty+ic} e^{-z^2} dz$$

for nonzero values of  $c$ .

(a) Let  $\rho > 0$  and consider the rectangular contour with corners at  $-\rho$ ,  $\rho$ ,  $\rho + ic$ , and  $-\rho + ic$ . If  $\gamma$  is the vertical edge from  $\rho$  to  $\rho + ic$ , prove that

$$\left| \int_{\gamma} e^{-z^2} dz \right| \leq ce^{c^2} e^{-\rho^2}.$$

Prove the same bound for the integral along the other vertical edge (from  $-\rho + ic$  to  $-\rho$ ).

(b) By sending  $\rho \rightarrow \infty$  in part (a), use (1) to find a formula for

$$\int_{-\infty+ic}^{\infty+ic} e^{-z^2} dz$$

when  $c$  is arbitrary.

Textbook exercises: 8.2.1(abd), 8.2.3(ac), 8.2.4, 8.2.6(ac), 8.2.7, 8.2.8.

For 8.2.1, you only need to express the inversion formula, not prove it. For 8.2.6, you just need to write  $f(t)$  as an integral, as in equation (14) on page 470. You will need to use the extra problem to solve 8.2.1(b) and 8.2.7.