8.3.1 a) $F(t) = 3\cos(2t) - 8e^{-2t}$

From table of laplace transforms on pg. 479 we have

$$\mathcal{L}\{\cos(\omega t)\}(s) = \frac{s}{s^2+\omega^2} \qquad Re(s) > 0$$

$$\mathcal{L}\{e^{at}\}(s) = \frac{1}{s-a} \qquad Re(s) > Re(a)$$

So we get:

$$\mathcal{L}\{F\}(s) = 3 \cdot \frac{s}{s^2+4} - 8 \cdot \frac{1}{s+2} \qquad Re(s) > 0.$$

$$= \frac{3s}{s^2+4} - \frac{8}{s+2}.$$

d) $F(t) = \begin{cases} 0, & t < 1 \\ 1, & 1 \le t \le 2 \\ 0, & 2 < t \end{cases}$

$$\mathcal{L}\{F\}(s) = \int_0^\infty F(t)\, e^{-st}\, dt$$

$$= \int_1^2 e^{-st}\, dt = \left.\frac{e^{-st}}{-s}\right|_1^2 = \frac{e^{-2s} - e^{-s}}{-s}$$

$$= \frac{e^{-s} - e^{-2s}}{s} \qquad Re(s) > 0.$$

e) $F(t) = \sin^2(t) = \dfrac{1-\cos(2t)}{2}$

So $\mathcal{L}\{F\}(s) = \dfrac{1}{2s} - \dfrac{1}{2} \cdot \dfrac{s}{s^2+4}$ using $\mathcal{L}\{1\}(s) = \dfrac{1}{s}$ $Re(s) > 0$.

8.3.3 a) $$\frac{1}{s^2+4} = \frac{1}{2}\cdot\frac{2}{s^2+4} \quad \left(= \frac{1}{2}\frac{\omega}{s^2+\omega^2}\right)$$

is the L.T. of $f(t) = \frac{1}{2}\sin(2t)$

b) $$\frac{4}{(s-1)^2} = \mathcal{L}\{4te^t\}(s)$$

since $\mathcal{L}\{t^n e^{at}\} = \frac{n!}{(s-a)^{n+1}}$

c) $$\frac{s+1}{s^2+4s+4} = \frac{s+1}{(s+2)^2} = \frac{s+2}{(s+2)^2} - \frac{1}{(s+2)^2}$$

$$= \frac{1}{s+2} - \frac{1}{(s+2)^2}$$

$$= \mathcal{L}\{e^{-2t}\}(s) - \mathcal{L}\{te^{-2t}\}(s)$$

$$= \mathcal{L}\{e^{-2t} - te^{-2t}\}(s).$$

8.3.4: $$f_\tau(t) = \begin{cases} 0, & 0 \le t < \tau \\ f(t-\tau), & \tau \le t < \infty \end{cases}$$

$$\mathcal{L}\{f_\tau(t)\}(s) = \int_0^\infty f_\tau(t)e^{-st}\,dt$$

$$= \int_\tau^\infty f(t-\tau)e^{-st}\,dt \qquad \text{let } \theta = t-\tau, \quad d\theta = dt$$

$$= \int_0^\infty f(\theta)e^{-s(\theta+\tau)}\,d\theta$$

$$= e^{-s\tau}\int_0^\infty f(\theta)e^{-s\theta}\,d\theta = e^{-s\tau}\mathcal{L}\{f(t)\}(s)$$

$\boxtimes$

8.3.5 a): $f' - f = e^{3t}$, $\quad f(0) = 3$

Applying Laplace transform to b/s we get:

$$\mathcal{L}\{f'\} - \mathcal{L}\{f\} = \mathcal{L}\{e^{3t}\}$$

$$\Rightarrow \left(s\,\mathcal{L}\{f\}(s) - f(0)\right) - \mathcal{L}\{f\}(s) = \frac{1}{s-3} \qquad \text{Let } \mathcal{L}\{f\} = g.$$

$$\Rightarrow \quad s\,g(s) - 3 - g(s) = \frac{1}{s-3}$$

$$\Rightarrow \quad g(s) = \frac{1}{s-1}\left(3 + \frac{1}{s-3}\right)$$

$$= \frac{3}{s-1} + \frac{1}{(s-1)(s-3)}$$

$$\cancel{= \frac{3}{s-1} + \frac{1}{s^2-4s+3}}$$

$$\cancel{= \frac{3}{s-1} + \frac{1}{(s-2)^2-1}}$$

$$g(s) = \frac{3}{s-1} + \frac{1}{2}\cdot\frac{1}{s-3} - \frac{1}{2}\cdot\frac{1}{s-1} \qquad \text{(partial fractions)}$$

$$\mathcal{L}\{f\}(s) = g(s) = \frac{5}{2}\cdot\frac{1}{s-1} + \frac{1}{2}\cdot\frac{1}{s-3}$$

So $f(t) = \frac{5}{2}e^{t} + \frac{1}{2}e^{3t}$ solves it.

c) $f'' - f' - 2f = e^{-t}\sin(2t)$, $\quad f(0) = 0$, $f'(0) = 2$

Let $\mathcal{L}\{f\}(s) = g(s)$.

Using eq. 8 on page 478 we get:

$$\left(s^2 g(s) - s f(0) - f'(0)\right) - \left(s g(s) - f(0)\right) - 2g(s) = \mathcal{L}\{e^{-t}\sin(2t)\}(s)$$

$$\Rightarrow \left(s^2 g(s) - 2\right) - \left(s g(s)\right) - 2g(s) = \frac{2}{(s+1)^2+4}$$

$$\Rightarrow \quad (s^2 - s - 2)\, g(s) \quad - 2 \quad = \frac{2}{(s+1)^2+4}$$

$$\Rightarrow \qquad g(s) = \left(\frac{2}{(s+1)^2+4} + 2\right)\frac{1}{s^2-s-2}$$

Now $\dfrac{2}{\left((s+1)^2+4\right)(s^2-s-2)} = \dfrac{3s-1}{26\left((s+1)^2+4\right)} + \dfrac{2}{39}\cdot\dfrac{1}{s-2} - \dfrac{1}{6}\cdot\dfrac{1}{s+1}$ (partial fractions)

And $\dfrac{2}{s^2-s-2} = \dfrac{2}{(s-2)(s+1)} = \dfrac{2}{3}\cdot\dfrac{1}{s-2} - \dfrac{2}{3}\cdot\dfrac{1}{s+1}$.

So $$g(s) = \frac{28}{39}\cdot\frac{1}{s-2} - \frac{5}{6}\cdot\frac{1}{s+1} + \frac{3s-1}{26\left((s+1)^2+4\right)}$$

$$= \frac{28}{39}\cdot\frac{1}{s-2} - \frac{5}{6}\cdot\frac{1}{s+1} + \frac{3(s+1)}{26\left((s+1)^2+4\right)} - \frac{4}{26\left((s+1)^2+4\right)}$$

$$= \frac{28}{39}\cdot\frac{1}{s-2} - \frac{5}{6}\cdot\frac{1}{s+1} + \frac{3}{26}\cdot\frac{s+1}{(s+1)^2+4} - \frac{1}{13}\cdot\frac{2}{(s+1)^2+4}$$

So $$f(t) = \frac{28}{39}e^{2t} - \frac{5}{6}e^{-t} + \frac{3}{26}e^{-t}\cos(2t) - \frac{1}{13}e^{-t}\sin(2t)$$

8.3.6: $F(t) = e^{-t}$ so $\mathcal{L}\{F\}(s) = \dfrac{1}{s+1}$ $\quad Re(s) > -1$

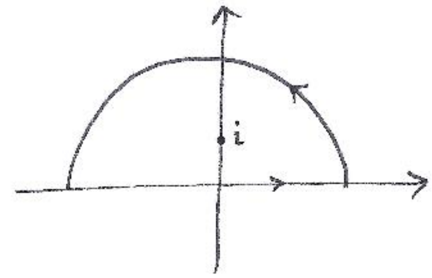
The inversion formula states $$F(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{i\omega+1}e^{i\omega t}\,d\omega$$

i.e. $$F(t) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{1}{\omega-i}e^{i\omega t}\,d\omega$$

Since $t > 0$, Jordan's lemma applies with an upper semicircular contour to yield:

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{e^{i\omega t}}{\omega - i}\,d\omega = \operatorname{Res}(i)$$

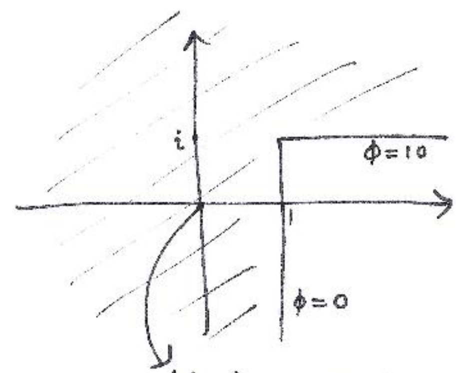
$$= e^{iit} = e^{-t} = F(t)$$



$\boxed{//}$

3.4.2: Let $\arg_0(z)$ denote the argument function with values between 0 and 2π.

Then $\phi = A \arg_0(z-(1+i)) + B$ is harmonic on the shaded region.



Boundary conditions give:

$A\cdot 0 + B = 10 \quad \Rightarrow B = 10$

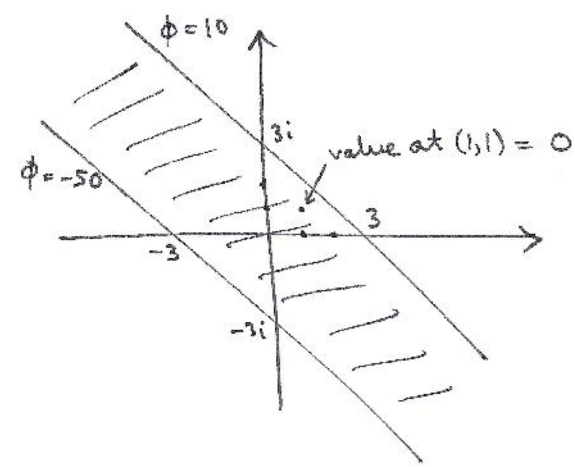
$A\cdot \frac{3\pi}{2} + B = 0 \quad \Longrightarrow A = -\frac{10\cdot 2}{3\pi} = \frac{-20}{3\pi}.$

$\phi(0,0) = -\frac{20}{3\pi}\cdot\frac{5\pi}{4} + 10$

$= 10 - \frac{25}{3} = \boxed{\frac{5}{3}}$

So $\phi = -\frac{20}{3\pi}\arg_0(z-(1+i)) + 10$ is the harmonic function on the shaded region with the given boundary conditions.

3.4.3: The function $\operatorname{Re}(z) = x$ is constant along vertical lines. Multiplication by $e^{\frac{-i\pi}{4}}$ turns the region into a vertical one.



So set $\phi = A\operatorname{Re}((1-i)z) + B$

Note $\operatorname{Re}((1-i)z) = x+y$.

So $\phi(x,y) = A(x+y)+B$

Boundary conditions give: $3A+B=10$

$-3A+B=-50$

$\Rightarrow B=-50+3A$

$\Rightarrow 3A-50+3A=10 \qquad A=10, \; B=-20$

Thus $\phi(x,y) = 10(x+y)-20$

Value at $(1,1)$ is $\boxed{0.}$

4.7.1: $D: |z|<1$ the unit disk.

ϕ harmonic on D with $\phi(z) \geq -5 \quad \forall z \in D$

and $\phi(\frac{i}{2}) = -5$.

Then ϕ achieves its minimum value in the interior of D.

Therefore ϕ is constant in D by the maximum-minimum principle.

i.e. $\phi(z)=-5$ is the only such function.

4.7.4: The maximum as well as the minimum temperatures are always attained on the boundary of the region in steady state heat flow. i.e. after long enough time, the hottest and the coldest points are on the boundary of the solid.

4.7.11: The temperature on the unit circle is maintained as shown. By Thm 30, (pg. 224), the temperature at the center is given by $u(0) = \frac{1}{2\pi}\int_0^{2\pi} u(e^{it})\,dt$ is the average, i.e. $u(0) = \boxed{3.}$