MATH 4220: HOMEWORK 2 SOLUTIONS 9/19/2015

§1.7 #2: Recall the formula for stereographic projection:

The point $z = x+iy$ in the plane is mapped to

$$(x_1, x_2, x_3) = \frac{1}{|z|^2+1}\left(2\,\mathrm{Re}\, z,\ 2\,\mathrm{Im}\, z,\ |z|^2-1\right) \text{ on the sphere.}$$

(a) $\frac{1}{|z|^2}z = \frac{1}{\bar z} = \frac{x+iy}{x^2+y^2}$ and $\left|\frac{1}{\bar z}\right| = \frac{1}{|z|}$, so we get

Let (y_1, y_2, y_3) be the point on the sphere that projects to $1/\bar z$.

Then $$(y_1, y_2, y_3) = \frac{1}{1/|z|^2+1}\left(\frac{2x}{x^2+y^2},\ \frac{2y}{x^2+y^2},\ \frac{1}{|z|^2}-1\right)$$

$$= \frac{|z|^2}{1+|z|^2}\left(\frac{2\,\mathrm{Re}\, z}{|z|^2},\ \frac{2\,\mathrm{Im}\, z}{|z|^2},\ \frac{1-|z|^2}{|z|^2}\right)$$

$$= \frac{1}{1+|z|^2}\left(2\,\mathrm{Re}\, z,\ 2\,\mathrm{Im}\, z,\ 1-|z|^2\right)$$

So, $(y_1, y_2, y_3) = (x_1, x_2, -x_3) =$ projection of z.

Since x_1x_2-plane is the equatorial plane, we see the two points are reflections in the equatorial plane.

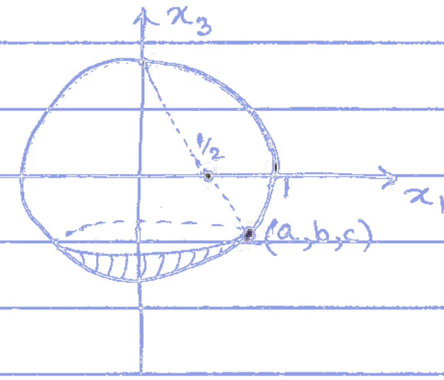
(b) $-\frac{1}{\bar z}$ has the same absolute value, so the third coordinate will be the same as that of $1/\bar z$.

But the real and imaginary parts are just negated, so y_1, y_2 will be the negatives of those for $1/\bar z$.

Thus $-1/\bar z$ corresponds to the point $(-x_1, -x_2, -x_3)$ on the sphere, which is diametrically opposite to (x_1, x_2, x_3).

#5: (a) The "right" half plane $\{z \mid \mathrm{Re}\, z > 0\}$ maps to the "right" half sphere $\boxed{\{\vec{x} \in S^2 \mid x_1 > 0\}}$

(b) The disk $\{z \mid |z| < 1/2\}$ maps to a polar cap around the South Pole.



Let the image of $z = 1/2$ be (a,b,c). Then it is given by

$$(a,b,c) = \frac{1}{\frac{1}{4}+1}\left(1, 0, \tfrac{1}{4}-1\right)$$

$$= \frac{4}{5}\left(1, 0, -3/4\right) = \left(\tfrac{4}{5}, 0, -3/5\right)$$

Thus it is the polar cap given by $\boxed{\{\vec{x} \in S^2 \mid x_3 < -3/5\}}$.

(c) The annulus $\{z \mid 1 < |z| < 2\}$ gets mapped to another annulus on the sphere. $\{|z| = 1\}$ gets mapped to the equator of S^2, the sphere. $z = 2$ gets mapped to $\frac{1}{4+1}(4, 0, 4-1) = (4/5, 0, 3/5)$.

So the image on the sphere is $\boxed{\{\vec{x} \in S^2 \mid 0 < x_3 < 3/5\}}$.

(d) $\{z \mid |z| > 3\}$ gets mapped to a northern polar cap.

$z = 3$ goes to $\frac{1}{9+1}(6, 0, 9-1) = \left(\tfrac{3}{5}, 0, 4/5\right)$

So the cap we get is $\boxed{\{\vec{x} \in S^2 \mid x_3 > 4/5\}}$.

(e) The line $y = x$ corresponds to z with the same real and imaginary parts which thus implies there images have the same x_1 & x_2 coordinates. So the image is the great circle given by $S^2 \cap \{x_1 = x_2\}$

$= \boxed{\{\vec{x} \in S^2 \mid x_1 = x_2\}}$.

EXTRA : $Z=(x_1,x_2,x_3)$ corresponds to z and let
$W=(\hat{x}_1,\hat{x}_2,\hat{x}_3)$ correspond to $w=1/z$.

We know by equation (1) (pg. 46) that

$$Z=\frac{1}{|z|^2+1}\left(2\operatorname{Re} z, 2\operatorname{Im} z, |z|^2-1\right)$$

Similarly, $$W=\frac{1}{\frac{1}{|z|^2}+1}\left(2\operatorname{Re}\tfrac{1}{z}, 2\operatorname{Im}\tfrac{1}{z}, \tfrac{1}{|z|^2}-1\right)$$

i.e. $$W=\frac{|z|^2}{1+|z|^2}\left(\frac{2\operatorname{Re} z}{|z|^2}, \frac{-2\operatorname{Im} z}{|z|^2}, \frac{1-|z|^2}{|z|^2}\right)$$

$$=\frac{1}{1+|z|^2}\left(2\operatorname{Re} z, -2\operatorname{Im} z, -(|z|^2-1)\right)$$

Thus $\hat{x}_1=x_1$, $\hat{x}_2=-x_2$, $\hat{x}_3=-x_3$, i.e. the map $1/z$ corresponds to rotation by π about the x_1-axis.

§2.1 #4: (a) $w=f(z)=1/z$ Consider the image of the circle $|z|=r$.
It is parametrized by $\gamma(t)=r\cos t+ir\sin t=re^{it}$
Then $w=1/z$ is parametrized by

$$\frac{1}{\gamma(t)}=\frac{1}{r(\cos t+i\sin t)}=\frac{1}{r}(\cos t-i\sin t)=\frac{e^{-it}}{r}$$

is $|w|=\frac{1}{r}$, the circle of radius $1/r$ (going "the other" way!)

(b) The ray $\operatorname{Arg} z=\theta_0$ $-\pi<\theta_0<\pi$ is parametrized by
$z=te^{i\theta_0}$ $0<t<\infty$.

So $\frac{1}{z}=\frac{e^{-\theta_0}}{t}$ & $1/t$ also goes from 0 to ∞ (backwards).

i.e. the image is the ray $\operatorname{Arg} w=-\theta_0$.

(c) The circle $|z-1| = 1$.

Using $w = 1/z$, we get

$$\left| 1/w - 1 \right| = 1$$

$$\Rightarrow \quad |1-w| = |w|$$

This is the line of points whose distance to 1 is the same as their distance to 0, i.e. the perpendicular bisector to the segment $[0,1] \subseteq \mathbb{R}$.

So it is the line $\operatorname{Re} z = 1/2$.

[You can also parametrize the circle as $z = 1 + e^{it}$, & see that

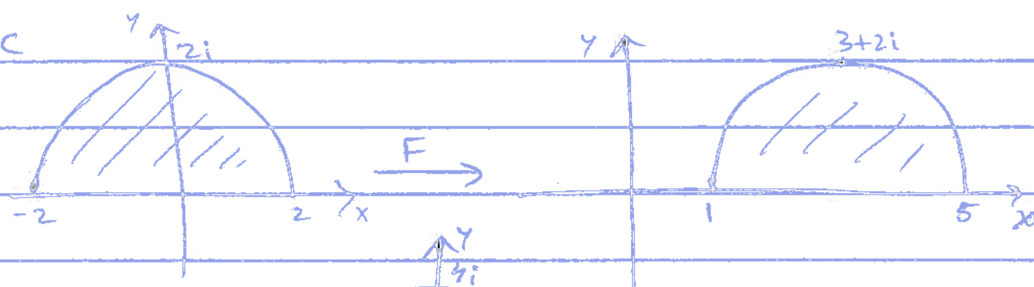
$$\frac{1}{z} = \frac{1}{1+e^{it}} = \frac{1+e^{-it}}{(1+\cos t)^2 + \sin^2 t}$$

$$= \frac{(1+\cos t) - i \sin t}{2(1+\cos t)} = \left(\tfrac{1}{2} + \tfrac{i}{2} \frac{\sin t}{1+\cos t} \right)$$

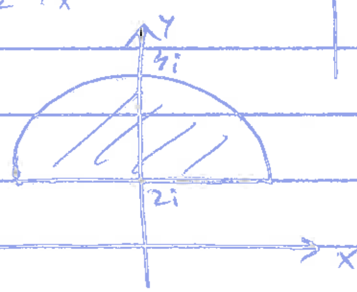
$$= \tfrac{1}{2} + \tfrac{i}{2} \tan(t/2)$$ to check that it's onto.].

#7: $F(z) = z + c$

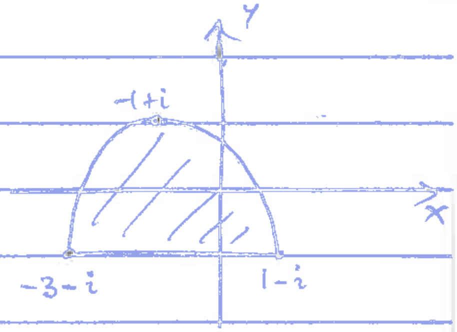
(a) $c = 3$:



(b) $c = 2i$: The image is



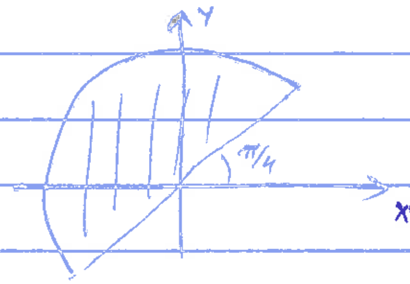
(c) $c = -1-i$: The image is



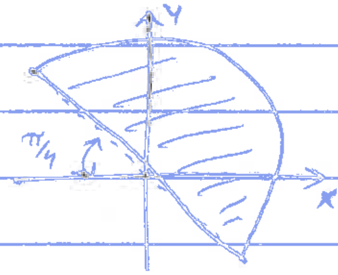
#8: $G(z) = e^{i\phi} z$, $\phi \in \mathbb{R}$

(a) $\phi = \pi/4$:

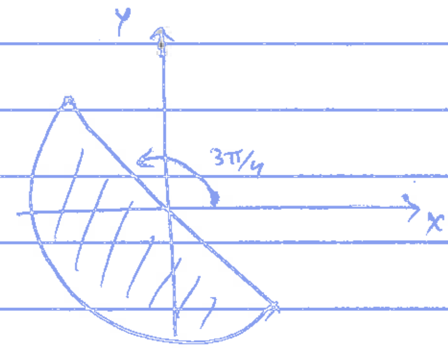
The image of the semi-disk is



(b) $\phi = -\pi/4$:

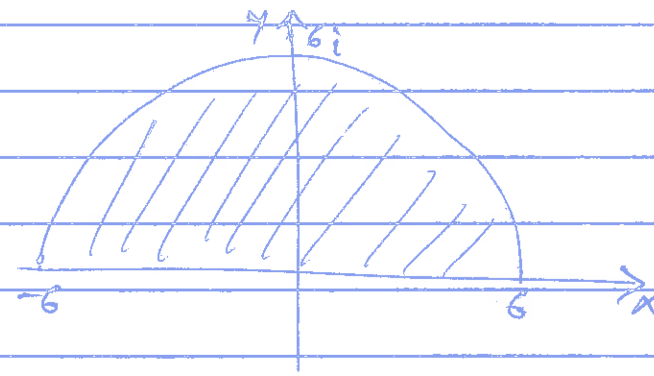


(c) $\phi = 3\pi/4$: ⟶

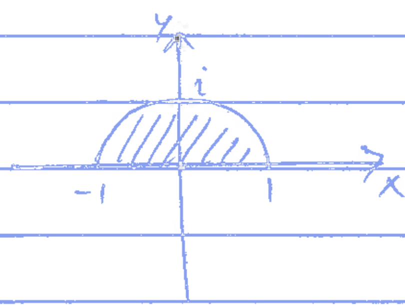


#9: $H(z) = \rho z$, $\rho \in R^{>0}$

(a) $\rho = 3$; the image is



(b) $\rho = 1/2$:



EXTRA: COMPUTE LIMITS:

(a) $\displaystyle\lim_{z\to 2i} \frac{z^2+4}{z-2i} = \lim_{z\to 2i} \frac{(z+2i)\cancel{(z-2i)}}{\cancel{(z-2i)}} = \boxed{4i}$

(b) $\displaystyle\lim_{z\to 1+i} \frac{z^2}{z+5} = \frac{(1+i)^2}{(1+i)+5} = \frac{2i}{6+i} = \frac{2i(6-i)}{37} = \boxed{\frac{2+12i}{37}}$

(c) $\displaystyle\lim_{z\to 1-i} \frac{z-1}{z^2+2i} = \lim_{z\to(1-i)} \frac{z-1 \neq 0}{(z-(1-i))(z+(1-i))} = \boxed{\infty}$

(d) $\displaystyle\lim_{z\to\infty} z^3+2z^2-4iz = \boxed{\infty}$

(e) $\lim_{z\to\infty} \frac{iz^2+(1-3i)z+(-5-2i)}{3z^2+(4+i)z+(2-3i)}$

$= \lim_{z\to\infty} \frac{i+(1-3i)/z+(-5-2i)/z^2}{3+(4+i)/z+(2-3i)/z^2} = \boxed{i/3}$

§2.3: #2: $f(z)$ is differentiable at z_0.

Define $\lambda(z) = \frac{f(z)-f(z_0)}{z-z_0} - f'(z_0)$ for z near z_0.

and $\lambda(z_0) = 0$.

Then the required formula is satisfied: i.e.

$f(z) = f(z_0) + f'(z_0)(z-z_0) + \lambda(z)(z-z_0)$.

So we only need to show $\lim_{z\to z_0} \lambda(z) = 0$ (i.e λ is continuous)

But by definition, $\lim_{z\to z_0}\lambda(z) = \lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0} - \lim_{z\to z_0} f'(z_0)$

$= f'(z_0) - f'(z_0) = 0$. ☒

#8: Say f is analytic at z_0 and $f'(z_0) \neq 0$.

The absolute value function $|\cdot| : \mathbb{C} \to \mathbb{R}$ is continuous, and continuous functions commute with limits, so;

$$\left| \lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0} \right| = \lim_{z\to z_0} \left| \frac{f(z)-f(z_0)}{z-z_0} \right|$$

$$\Rightarrow \quad |f'(z_0)| = \lim_{z\to z_0} \frac{|f(z)-f(z_0)|}{|z-z_0|}.$$

Similarly, if $f'(z_0)\neq 0$, then in some neighborhood of z_0 $f'(z)$ is not zero, hence the argument function can be continuously defined.

Restricting the limit to z in that neighborhood of z_0, we get:

$$\arg\left(\lim_{z\to z_0} \frac{f(z)-f(z_0)}{(z-z_0)}\right) = \lim_{z\to z_0} \arg\left(\frac{f(z)-f(z_0)}{z-z_0}\right)$$

$$\Rightarrow \arg f'(z_0) = \lim_{z\to z_0}\left[\arg(f(z)-f(z_0)) - \arg(z-z_0)\right].$$

$\blacksquare$

#11: (a) $8\bar{z}+i=w$ is $\boxed{\text{not analytic}}$ anywhere, because if it were, $\frac{w-i}{8}=\bar{z}$ would also be.

(b) $\frac{z}{\bar{z}+2}=w$ is $\boxed{\text{not analytic}}$ anywhere,

because if it were, so would $\frac{z}{w}-2=\bar{z}$.

(c) $\frac{z^3+2z+i}{z-5}$ is $\boxed{\text{analytic}}$ everywhere $\boxed{\text{except at } z=5}$

by Thm 3 (page 69) and the fact that z is analytic.

At $z=5$, it is not continuous.

(h) $\frac{|z|+z}{2}=w$ is $\boxed{\text{not analytic}}$ anywhere, because if it were, $-z+2w=|z|$ would be too.

(i) $(1-2i)x+(2+i)y = (1-2i)\frac{(z+\bar{z})}{2} + (2+i)\frac{(z-\bar{z})}{2i}$

$$= \frac{1}{2}\Big((1-2i)(z+\bar{z}) - i(2+i)(z-\bar{z})\Big) = (1-2i)z$$

is $\boxed{\text{analytic}}$ everywhere.

EXTRA: $f(z)=1/z$ $\quad f'(z_0) = \lim_{z\to z_0} \frac{\frac{1}{z}-\frac{1}{z_0}}{z-z_0} = \lim_{z\to z_0} \frac{\left(\frac{z_0-z}{z_0 z}\right)}{z-z_0}$

$$= \lim_{z\to z_0} \frac{-1}{z_0 z} = \boxed{-1/z^2}$$