

Math 4220: Homework 2

Due Thursday, September 10 in class.

1.7. For these problems it will help to know the algebraic formulas for the stereographic projection, which are given in the book.

Textbook exercises: 1.7.2, 1.7.5.

Extra problem: Suppose $w = 1/z$. Let $Z = (x_1, x_2, x_3)$ be the point on the sphere corresponding to z under the stereographic projection, and let $W = (\hat{x}_1, \hat{x}_2, \hat{x}_3)$ be the point on the sphere corresponding to w . Use the formulas for (x_1, x_2, x_3) and $(\hat{x}_1, \hat{x}_2, \hat{x}_3)$ to prove that $\hat{x}_1 = x_1$, $\hat{x}_2 = -x_2$, and $\hat{x}_3 = -x_3$.

2.1. Textbook exercises: 2.1.4, 2.1.7, 2.1.8, 2.1.9. For 4(c) there are two possible arguments. First, if $|z - 1| = 1$ you can use direct computation to prove that $\operatorname{Re}(1/z) = 1/2$. Second, you can argue that the circle $|z - 1| = 1$ passes through the origin, so based on our work in class, its image under the inversion map $z \mapsto 1/z$ must be a line. To figure out which line, simply choose two points z_1, z_2 on the circle and find their images $1/z_1, 1/z_2$. There is only one line that passes through the points $1/z_1$ and $1/z_2$.

2.2. You may use the following result. Suppose the task is to compute

$$\lim_{z \rightarrow z_0} \frac{f(z)}{g(z)}.$$

If $\lim_{z \rightarrow z_0} g(z) = 0$ and $\lim_{z \rightarrow z_0} f(z) = c \neq 0$, then $\lim_{z \rightarrow z_0} f(z)/g(z) = \infty$. However, if $\lim_{z \rightarrow z_0} g(z) = 0$ and also $\lim_{z \rightarrow z_0} f(z) = 0$, the limit of the fraction is an “indeterminate form” (in the language of single-variable calculus) and you have to use other techniques to compute it. Those other techniques could be clever simplification (e.g. canceling a common factor from the numerator and denominator), L’Hôpital’s Rule, or series expansion.

Also, a trick for evaluating $\lim_{z \rightarrow \infty} \frac{f(z)}{g(z)}$ when both $f(z)$ and $g(z)$ are polynomials is to find the highest power of z that appears anywhere and then divide both the numerator and denominator by that power of z .

No textbook exercises from this section. Extra problem: Compute each of the following limits.

(a) $\lim_{z \rightarrow 2i} \frac{z^2 + 4}{z - 2i}$

(b) $\lim_{z \rightarrow 1+i} \frac{z^2}{z + 5}$

(c) $\lim_{z \rightarrow 1-i} \frac{z - 1}{z^2 + 2i}$

(d) $\lim_{z \rightarrow \infty} z^3 + 2z^2 - 4iz$

(e) $\lim_{z \rightarrow \infty} \frac{iz^2 + (1 - 3i)z + (-5 - 2i)}{3z^2 + (4 + i)z + (2 - 3i)}$

2.3. You may use the result of textbook exercise 2.3.1.

Textbook exercises: 2.3.2, 2.3.8, 2.3.11(abch).

Mark the following problem as 2.3.11(i): Determine whether the function $z = x + iy \mapsto (1 - 2i)x + (2 + i)y$ is analytic. Hint: Use the formulas $x = (z + \bar{z})/2$ and $y = (z - \bar{z})/2i$.

Extra problem: Use the definition of the derivative (i.e. limits) to compute the derivative of $f(z) = 1/z$.