MATH 4220 HOMEWORK 3 SOLUTIONS

2.4.1: (a) $w = \bar{z} = x - iy$

$$\frac{\partial u}{\partial x} = 1 \neq \frac{\partial v}{\partial y} = -1$$ so not ~~analytic~~ even differentiable anywhere.

(b) $w = \mathrm{Re}(z) = x$

$$\frac{\partial u}{\partial x} = 1 \neq \frac{\partial v}{\partial y} = 0$$ so also nowhere differentiable.

(c) $w = 2y - ix \quad (u = 2y,\ v = -x)$

$$\frac{\partial u}{\partial x} = 0 = \frac{\partial v}{\partial y}$$

but $-\frac{\partial v}{\partial x} = 1 \neq \frac{\partial u}{\partial y} = 2$ so nowhere differentiable.

2.4.2: $h(z) = x^3 + 3xy^2 - 3x + i(y^3 + 3x^2y - 3y)$

Here $\frac{\partial u}{\partial x} = 3x^2 + 3y^2 - 3$, $\frac{\partial v}{\partial y} = 3y^2 + 3x^2 - 3$ (always equal)

and $-\frac{\partial u}{\partial y} = -(6xy)$, $\frac{\partial v}{\partial x} = 6xy$

So $-\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$ only when $x = 0$ or $y = 0$ (i.e. on the coordinate axes)

So h is only differentiable on the coordinate axes, and thus on no open disk, so it is nowhere analytic.

2.4.8: f is analytic in a domain D and (say) $\mathrm{Re}\, f(z)$ is constant.

i.e. $u(z)$ is constant, so $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0$

By analyticity, $\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 0$ & $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 0$ on D.

So $\mathrm{Im}\, f(z) = v(z)$ is constant (by Thm 1 pg. 40), on D.

Thus $f(z) = Re(f) + i\, Im(f)$ is a constant function, on D.
The same proof works if $Im\, f(z)$ is constant. $\square$

2.4.10: $f(z)$ analytic and real valued in D.

Then $Im\, f(z) \equiv 0$ on D (thus constant).
So by the previous problem, $f(z)$ is constant on D.

2.5.1: (a) $f(z) = z^2 + 2z + 1$

$Re\, f(z) = u = (x^2 - y^2) + 2x + 1 = (x+1)^2 - y^2$

$Im\, f(z) = v = 2xy + 2y$

Both are C^2 (i.e second order partials are continuous).

$\nabla^2 u = 2 - 2 = 0.$ u harmonic

$\nabla^2 v = \frac{\partial}{\partial x}(2y) + \frac{\partial}{\partial y}(2x+2) = 0 + 0 = 0,$ v harmonic.

(b) $g(z) = 1/z = (x - iy)/(x^2+y^2)$ $z \neq 0$

$u = \frac{x}{x^2+y^2}$, $v = \frac{-y}{x^2+y^2}$ C^2 for $z \neq 0$

$$\frac{\partial u}{\partial x} = \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{(x^2+y^2)^2(-2x) - 2(y^2-x^2)(x^2+y^2)(2x)}{(x^2+y^2)^4} = \frac{4x(x^2-y^2) - 2x(x^2+y^2)}{(x^2+y^2)^3}$$

$$\frac{\partial u}{\partial y} = \frac{-2xy}{(x^2+y^2)^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{(x^2+y^2)^2(-2x) + 2xy(x^2+y^2)(4y)}{(x^2+y^2)^4} = \frac{8xy^2 - 2x^3 - 2xy^2}{(x^2+y^2)^3}$$

And indeed $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$

Similarly for $v = -y/(x^2+y^2)$,

$$\frac{\partial v}{\partial x} = \frac{2xy}{(x^2+y^2)^2}$$

$$\frac{\partial^2 v}{\partial x^2} = \frac{(x^2+y^2)^2(2y) - 2xy\cdot 2(x^2+y^2)(2x)}{(x^2+y^2)^4} = \frac{2yx^2+2y^3-8yx^2}{(x^2+y^2)^3} = \frac{2y(y^2-3x^2)}{(x^2+y^2)^3}$$

$$\frac{\partial v}{\partial y} = \frac{-(x^2+y^2)+y(2y)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

& $$\frac{\partial^2 v}{\partial y^2} = \frac{(x^2+y^2)^2(2y) - (y^2-x^2)2(x^2+y^2)(2y)}{(x^2+y^2)^4} = \frac{2yx^2+2y^3-4y^3+4yx^2}{(x^2+y^2)^3} = \frac{2y(-y^2+3x^2)}{(x^2+y^2)^3}$$

And indeed $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$

(c) $h(z) = e^z = e^x\cos y + i e^x \sin y$

$u = e^x\cos y$, $v = e^x \sin y$ both C^2 everywhere.

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= e^x\cos y, & \frac{\partial^2 u}{\partial x^2} &= e^x\cos y. \\ \frac{\partial u}{\partial y} &= -e^x\sin y, & \frac{\partial^2 u}{\partial y^2} &= -e^x\cos y \end{aligned}\right\} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\left.\begin{aligned} \frac{\partial v}{\partial x} &= e^x\sin y, & \frac{\partial^2 v}{\partial x^2} &= e^x\sin y \\ \frac{\partial v}{\partial y} &= e^x\cos y, & \frac{\partial^2 v}{\partial y^2} &= -e^x\sin y \end{aligned}\right\} \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

Thus both u and v are harmonic.

2.5.2: $u = ax^2 + bxy + cy^2$ harmonic.

$\frac{\partial u}{\partial x} = 2ax + by$, $\frac{\partial^2 u}{\partial x^2} = 2a$

$\frac{\partial u}{\partial y} = bx + 2cy$, $\frac{\partial^2 u}{\partial y^2} = 2c$

Harmonic $\Rightarrow$ $2a + 2c = 0$, so $c = -a$.

Thus the most general harmonic polynomial of the form above is $a(x^2 - y^2) + bxy$.

2.5.3: (a) $u = y = Re(-iz)$ so it is harmonic on the whole plane.
It's harmonic conjugate is thus $Im(-iz) = Im(y - ix) = -x$.

[Alternatively check $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 + 0 = 0$, so harmonic,

then find v s.t. $\frac{\partial v}{\partial y} = 0$ and $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = -1$.

any v of the form $-x + c$ works, where $c \in \mathbb{R}$.]

(b) $u = e^x \sin y = Im(e^z)$, so u is harmonic everywhere.

Notice $u = e^x \sin y = Re(-ie^z)$,
so a harmonic conjugate for u is $Im(-ie^z) = -e^x \cos y$.

(c) $u = Im(e^{z^2})$ is harmonic everywhere by Thm 7 (pg 79).
As above, its harmonic conjugate is $-Re(e^{z^2})$

2.5.5: Given v is a harmonic conjugate for u, we know that $f = u + iv$ is an analytic function.

Then $-if = v - iu$ is also an analytic function, so $-u$ is a harmonic conjugate for v. ☒

2.5.3: u + v harmonic in domain D.

(a) Is $u+v$ harmonic?

Yes, $u+v$ is C^2 and

$$\frac{\partial^2(u+v)}{\partial x^2} + \frac{\partial^2(u+v)}{\partial y^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial y^2} = 0 + 0 = 0.$$

(b) Is uv harmonic?

No, take $u = x$, $v = x$, then u & v are harmonic, but $uv = x^2$ is not harmonic since,

$$\frac{\partial^2(x^2)}{\partial x^2} + \frac{\partial^2(x^2)}{\partial y^2} = 2 + 0 = 2 \neq 0.$$

(c) Is $\partial u/\partial x$ harmonic?

Yes, we compute,

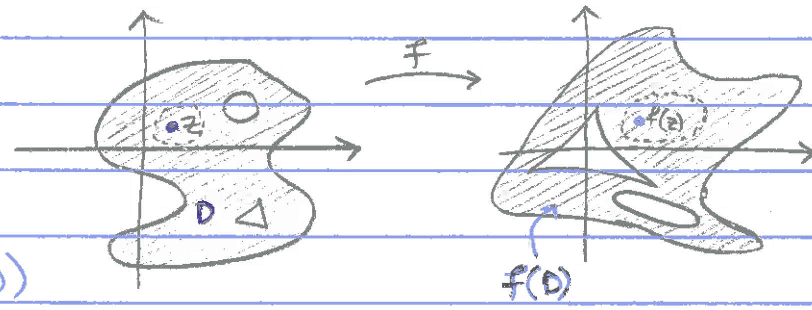
$$\frac{\partial^2}{\partial x^2}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^3 u}{\partial x^3} + \frac{\partial^3 u}{\partial y^2 \partial x}$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial x^2}\right) + \frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial y^2}\right)$$ since order of derivatives is interchangeable.

$$= \frac{\partial}{\partial x}\left(\nabla^2 u\right) = \frac{\partial}{\partial x}(0) = 0.$$ ☒

2.5.14: $f(z)$ analytic and $f(z) \neq 0$ for $z \in D$, a domain.
Prove $\ln|f(z)|$ is harmonic in D.

Since $f(z)$ is never 0, given $z \in D$, we can find a small disk D_0 around z, whose image under f is simply connected.
(e.g. by taking a component of the inverse image of a small disk around $f(z)$)



On this small disk around $f(z)$, the argument function can be continuously defined (since it does not contain 0 or go around 0).

Thus on this small disk around z, $\ln|f(z)|$ is just the real part of the analytic function

$$\ln(f(z)) = \ln|f(z)| + i \arg(f(z)) \quad \text{on } D_0 \subset D.$$

Hence $\ln|f(z)|$ is harmonic on D_0.
[Note: $\ln(f(z))$ is analytic being the composition of two analytic functions.]

Since this can be done for every $z \in D$, $\ln|f(z)|$ is harmonic on D.

[You can also just compute $\frac{\partial^2}{\partial x^2}(\ln|f(z)|) + \frac{\partial^2}{\partial y^2}(\ln|f(z)|)$ using $\ln|f(z)| = \frac{1}{2}\ln|f(z)|^2 = \frac{1}{2}\ln(u^2+v^2)$ and C-Riem. equations]

2.5.20: u harmonic on a disk. Show that u has a harmonic conjugate.

We want a function $v(x,y)$ on the disk s.t.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Say the disk is centered at (x_0, y_0). Following the hint, let's define $g(x,y)$ satisfying $\frac{\partial g}{\partial y} = \frac{\partial u}{\partial x}$.

i.e. define $g(x,y) = \int_{y_0}^{y} \frac{\partial u}{\partial x}\, dy$

Now consider $h(x,y) = \frac{\partial g}{\partial x} + \frac{\partial u}{\partial y}$. — eq. (1)

$$\frac{\partial h}{\partial y} = \frac{\partial^2 g}{\partial y \partial x} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 g}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2}$$

$$= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \quad \left(\text{since } \frac{\partial g}{\partial y} = \frac{\partial u}{\partial x}\right)$$

$$= 0 \quad (\text{since } u \text{ is harmonic}).$$

So $h(x,y)$ is independent of y,

i.e. $h(x,y) = h(x)$ for all (x,y) in the disk.

So we can define $v(x,y) = g(x,y) - \int_{x_0}^{x} h(x)\, dx$

Then check: $\frac{\partial v}{\partial y} = \frac{\partial g}{\partial y} = \frac{\partial u}{\partial x}$ (as $\int h\, dx$ is independent of y)

and $\frac{\partial v}{\partial x} = \frac{\partial g}{\partial x} - h(x)$ (by fundamental theorem of calculus)

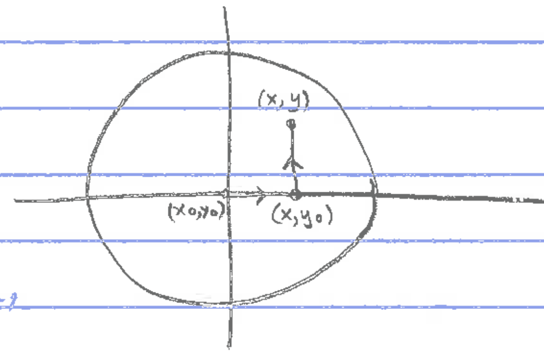
$\Rightarrow \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$ (by equation (1) above).

So v is a harmonic conjugate of u on the disk. ▨

EXTRA: We used the fact that the domain of u was a disk once when defining $g(x,y)$ by integrating along the straight line between (x,y_0) and (x,y) and once again when computing $\int_{x_0}^{x} h(x)\,dx$.

Both paths are in the domain ((x_0,y_0) to (x,y_0) and (x,y_0) to (x,y)) because the domain is a disk (hence convex).

No such trick would work for the punctured disk since some ~~any~~ path of straight lines would "fall into" the puncture, where u is not defined, (and could be infinite, as in exercise 2.5.21).



2.6.1: