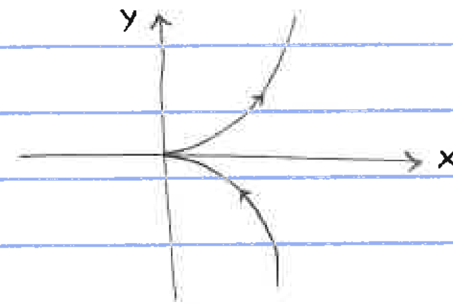
MATH 4220 HW4 SOLUTIONS

4.1.2: Show that $z'(t)$ never vanishes is necessary to ensure that smooth curves have no cusps.

Since we're trying to prove necessity, it is enough to provide a counterexample: Consider $z(t) = t^2 + it^3$, $-1 \le t \le 1$



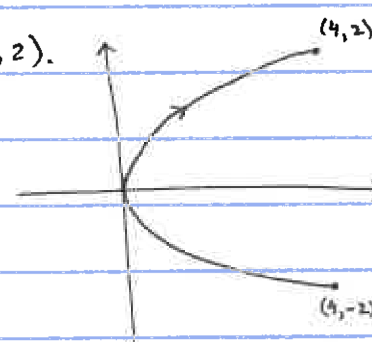
The curve satisfies $x = y^{2/3}$

the derivative $\frac{dx}{dy}$ approaches

$-\infty$ as $y \to 0^-$ but

$+\infty$ as $y \to 0^+$. So there is a cusp at $t=0$.

EXTRA 1: Parametrize $x = y^2$ from $(4,-2)$ to $(4,2)$.



let $y = t$, then $x = t^2$ & we get

$z(t) = t^2 + it$ $\quad -2 \le t \le 2$.

EXTRA 2: a) $z_0, z_1 \in \mathbb{C}$ fixed. $z : [0,1] \to \mathbb{C}$, $z(t) = (1-t)z_0 + t z_1$

This is the straight line segment from z_0 to z_1.

b) $w_0, w_1 \in \mathbb{C}$ fixed. Need a straight line segment from w_0 to $w_0 + w_1$.

One parametrization is given by $z(t) = w_0 + t w_1$ $\quad 0 \le t \le 1$.

4.2.3. a) $\int_0^1 (2t + it^2)\,dt = \left[t^2 + i\frac{t^3}{3}\right]_0^1 = 1 + \frac{i}{3}$

b) $\int_{-2}^{0} (1+i)\cos(it)\,dt = \frac{(1+i)}{i}\sin(it)\Big]_{-2}^{0} = \frac{1+i}{i}\left(-\sin(-2i)\right) = (1-i)\sin(2i)$

4.2.5: Evaluate $I = \int_C \left[\frac{6}{(z-i)^2} + \frac{2}{(z-i)} + 1 - 3(z-i)^2 \right] dz$, $C := |z-i| = 4$ counter-clockwise.

By example 2: $\int_C (z-i)^n dz = \begin{cases} 0 & n \neq -1 \\ 2\pi i & n = -1 \end{cases}$

So we get $I = 0 + 2\int_C (z-i)^{-1} dz + 0 + 0 = 2 \cdot 2\pi i = \boxed{4\pi i}$

4.2.6: Compute $\int_\Gamma \bar{z}\, dz$ where:

a) Γ is the circle $|z|=2$ once counterclockwise: $z(t) = 2e^{it}$ $0 \le t \le 2\pi$

$\int_\Gamma \bar{z}\, dz = \int_0^{2\pi} 2e^{-it} \cdot 2ie^{it}\, dt$ since $\bar{z}(t) = 2e^{-it}$ and $dz = 2ie^{it}dt$

$= 4i \int_0^{2\pi} dt = \boxed{8\pi i}$

b) Γ is the circle $|z|=2$ once clockwise: $z(t) = 2e^{-it}$ $0 \le t \le 2\pi$

So $\int_\Gamma \bar{z}\, dz = \int_0^{2\pi} 2e^{it} \cdot (-2ie^{-it})\, dt = -4i \int_0^{2\pi} dt = \boxed{-8\pi i}$

c) Γ is the circle $|z|=2$ traversed three times clockwise:

$z(t) = 2e^{-it}$ $0 \le t \le 6\pi$

So $\int_\Gamma \bar{z}\, dz = \int_0^{6\pi} 2e^{it}(-2ie^{-it})\, dt = -4i \int_0^{6\pi} dt = \boxed{-24\pi i}$

4.2.9: Evaluate $I = \int_\Gamma (x - 2xyi)\, dz$ over Γ: $z = t + it^2$, $0 \le t \le 1$.

$I = \int_0^1 (t - 2(t)(t^2)i) \cdot (1 + 2it)\, dt = \int_0^1 (t + 2it^2 - 2it^3 + 4t^4)\, dt$

$= \left[\frac{t^2}{2} + \frac{2it^3}{3} - \frac{2it^4}{4} + \frac{4t^5}{5} \right]_0^1 = 1/2 + 2i/3 - i/2 + 4/5 = \boxed{13/10 + 1/6\, i}$

4.2.14. a) C is the circle $|z|=3$. Want to bound $\left|\int_C \frac{dz}{z^2-i}\right|$.

First note that $f(z)=\dfrac{1}{z^2-i}$ is discontinuous only at $z=\pm\sqrt{i}$

neither of which is on C, so the theorem applies.
Length of C is 6π and by the triangle inequality,
$|z^2-i| \geqslant |z|^2-1 = 9-1 = 8$ on C.

By Thm 5 (pg. 170), we then have,

$$\left|\int_C \frac{dz}{z^2-i}\right| \leq \max_{z \text{ on } C}\left|\frac{1}{z^2-i}\right| \cdot \ell(C)$$

$$= \frac{1}{\min\limits_{z \text{ on } C}|z^2-i|} \cdot 6\pi = \frac{6\pi}{8} = \frac{3\pi}{4}.$$

b) γ is the vertical segment from R to $R+2\pi i$ (here $R>0$).
$f(z)=\dfrac{e^{3z}}{1+e^z}$ is discontinuous when $e^z=-1$,
i.e. for $z=(2n+1)\pi i$, none of which are on γ.

We have $|e^{3z}| = |e^{3\operatorname{Re} z}| = e^{3R}$ on γ
and $|1+e^z| \geqslant |e^z|-1 = e^R-1$ on γ.
Thus, $|f(z)| \leq \dfrac{e^{3R}}{e^R-1}$ on γ.

Finally, length of γ, $\ell(\gamma)=2\pi$, so we get

$$\left|\int_\gamma \frac{e^{3z}}{1+e^z}dz\right| \leq \frac{e^{3R}}{e^R-1}\cdot \ell(\gamma) = \frac{2\pi e^{3R}}{e^R-1}.$$

4.2.14. c) Γ is the arc of the circle $|z|=1$ in the first quadrant,

$$\Gamma: \quad z(t) = e^{it} \qquad 0 \le t \le \pi/2$$

$\text{Log}(z) = f(z)$ is discontinuous along the negative real axis, and Γ doesn't intersect that, so Thm 5 applies.

Since $\text{Log}(z(t)) = \ln 1 + i\,\text{Arg}(z(t)) = it$ on Γ,

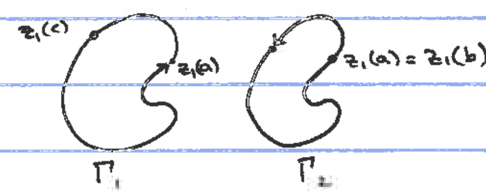
we have $|f(z)| = |\text{Log}(z)| \le \pi/2$ on Γ

Also $\ell(\Gamma) = \pi/2$. So we have:

$$\left|\int_\Gamma \text{Log}(z)\,dz\right| \le \max_{z \text{ on } \Gamma} |\text{Log}(z)| \cdot \ell(\Gamma) = \frac{\pi}{2}\cdot\frac{\pi}{4} = \pi^2/4$$

4.2.18. $\Gamma_1: \quad z = z_1(t) \qquad a \le t \le b$

$c \in (a,b)$



$$\Gamma_2: \quad z = z_2(t) = \begin{cases} z_1(t) & c \le t \le b \\ z_1(t-b+a) & b \le t \le b-a+c \end{cases}$$

f is continuous on the points of Γ_1 (hence also of Γ_2).

$$\int_{\Gamma_1} f(z)\,dz = \int_a^b f(z_1(t))\cdot z_1'(t)\,dt = \int_a^c f(z_1(t))\cdot z_1'(t)\,dt + \int_c^b f(z_1(t))\cdot z_1'(t)\,dt$$

And $$\int_{\Gamma_2} f(z)\,dz = \int_c^b f(z_2(t))\,z_2'(t)\,dt + \int_b^{b-a+c} f(z_2(t))\cdot z_2'(t)\,dt$$

$$= \int_c^b f(z_1(t))\cdot z_1'(t)\,dt + \int_b^{b-a+c} f(z_1(t-b+a))\cdot z_1'(t-b+a)\,dt$$

let $s = t-b+a$ in the second integral above, we have

$ds = dt$ and,

$$\int_{\Gamma_2} f(z)\,dz = \int_c^b f(z_1(t))\cdot z_1'(t)\,dt + \int_a^c f(z_1(s))\cdot z_1'(s)\,ds$$

$$= \int_a^b f(z_1(t))\cdot z_1'(t)\,dt = \int_{\Gamma_1} f(z)\,dz \qquad \blacksquare$$

4.3.4: If f is analytic at each point of a closed contour Γ, then $\int_\Gamma f(z)dz = 0$. $\boxed{\text{FALSE}}$

(e.g. $1/z$ is analytic at each point of $|z|=1$, but $\oint_{|z|=1} 1/z\,dz = 2\pi i$)

4.3.10: f continuous at the point z. Show:

$$\int_0^1 f(z+t\Delta z)\,dt \longrightarrow f(z) \quad \text{as } \Delta z \to 0.$$

Suppose we are given a (small) number $\varepsilon > 0$. We will show $\left|\int_0^1 f(z+t\Delta z)dt - f(z)\right|$ is smaller that ε when Δz is small enough.

Fix $\varepsilon>0$. By definition of continuity at z, we know there exists a number $\delta>0$, so that if $|\Delta z|<\delta$, we have

$$|f(z+\Delta z) - f(z)| < \varepsilon.$$

But if $0 \le t \le 1$, then $|\Delta z| < \delta \Rightarrow |t\Delta z| < \delta$, so we get

$$|f(z+t\Delta z) - f(z)| < \varepsilon.$$

Finally, note that $f(z) = \int_0^1 f(z)\,dt$. So we get

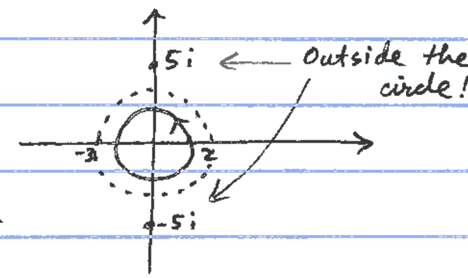
$$\left|\int_0^1 f(z+t\Delta z)dt - f(z)\right| \le \int_0^1 |f(z+t\Delta z) - f(z)|\,dt \le \int_0^1 \varepsilon\,dt = \varepsilon$$

when $|\Delta z| < \delta$.

So as $\Delta z \to 0$, the quantity above approaches 0, and hence $\int_0^1 f(z+t\Delta z)dt \to f(z)$. $\blacksquare$

4.4.10. a) $f(z) = \dfrac{z}{z^2+25}$ is analytic everywhere except at $z = \pm 5i$

and since $|z| \leq 2$ does not contain either $5i$ or $-5i$,
we can choose a simply connected domain (e.g. $|z| < 3$), in which f is analytic.



Then by the Cauchy Integral Theorem

$$\oint_{|z|=2} f(z)\,dz = 0$$

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b) $f(z) = e^{-z}(2z+1) = \dfrac{2z+1}{e^z}$ is analytic everywhere,

So by the Cauchy Integral Theorem $\oint_{|z|=2} f(z)\,dz = 0$.

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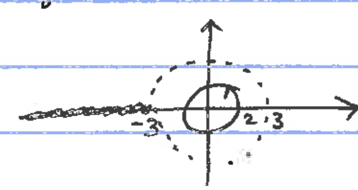
d) $f(z) = \operatorname{Log}(z+3)$ is analytic everywhere except on the part of the real axis to the left of -3.



So f is analytic on the simply connected domain $|z| < 3$, thus by the Cauchy Integral Theorem $\oint_{|z|=2} f(z)\,dz = 0$.

[In fact $\mathbb{C} \setminus \{t : t \leq -3\}$ is also simply connected and f is analytic on it, so no need to restrict to disk $|z| < 3$].

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