Math 4220: Hint for 5.1.20

Ordinary convergence means that for every z in the disk, the tail of the series

$$\sum_{j=N}^{\infty} z^j$$

has a limit of zero as $N \to \infty$. Uniform convergence on the disk means that the tail has a uniform bound that does not depend on z. To prove that the convergence is *not* uniform, you need to find $\varepsilon > 0$ such that no matter how large N is, there is still a value of z for which

$$\left| \sum_{j=N}^{\infty} z^j \right| \geq \varepsilon.$$

For each fixed z the series still converges, but there are values of z at which the series converges arbitrarily slowly. The overall series is $\sum_{j=0}^{\infty} z^j = \frac{1}{1-z}$, and the intuition is that the closer z gets to the pole at 1, the slower the series will converge.

Fortunately, since the sum is a geometric series, there is an exact formula

$$\sum_{j=N}^{\infty} z^j = z^N \sum_{k=0}^{\infty} z^k = \frac{z^N}{1-z}.$$

Notice that this sum *does* converge to 0 for fixed $|z| < 1$ as $N \to \infty$. When N is fixed, think about the real function

$$f(x) = \frac{x^N}{1-x}$$

for x between 0 and 1.