5.1.20: Uniform convergence of $\sum_{j=0}^{\infty} z^j$ to $\frac{1}{1-z}$ on $D = \{z \mid |z|<1\}$

means that for every $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ so that

$$\left| \sum_{j=0}^{n} z^j - \frac{1}{1-z} \right| < \varepsilon \quad \text{for any } z \in D \text{ and any } n \geq N.$$

To prove it is not uniformly convergent, we need to show

this is false, i.e. "there is an $\varepsilon = \varepsilon_0$ (say) so that

for every $N \in \mathbb{N}$ there is some $z \in D$ and $n \geq N$ for which

$$\left| \sum_{j=0}^{n} z^j - \frac{1}{1-z} \right| \geq \varepsilon_0 \text{"}.$$

Note that we know for a fixed $z \in D$ that $\frac{1}{1-z} = \sum_{j=0}^{\infty} z^j$, so that

$$\left| \sum_{j=0}^{N} z^j - \frac{1}{1-z} \right| = \left| \sum_{j=N+1}^{\infty} z^j \right| = |z|^{N+1} \left| \sum_{j=0}^{\infty} z^j \right| = \frac{|z|^{N+1}}{|1-z|} \geq |z|^{N+1}$$

We show that $\varepsilon_0 = 0.1$ suffices. Because no matter how big N is,

if we take $z = \sqrt[N+1]{0.2} \in \mathbb{R}$, then $z \in D$ and

$$\left| \sum_{j=0}^{N} z^j - \frac{1}{1-z} \right| \geq |z|^{N+1} = 0.2 > 0.1 = \varepsilon_0$$

(i.e. no single N works for all $z \in D$ to make the difference < 0.1)

So $\sum_{j=0}^{\infty} z^j$ does not <u>converge uniformly</u> to $\frac{1}{1-z}$ in the set D.

$\square$

5.1.21: 1^{st} step from $z=0$ to $z=1$.

2^{nd} step from $z=1$ of length $\frac{1}{2}$ in the direction of $e^{i\alpha}$.

So end up at $1+\frac{e^{i\alpha}}{2}$.

3^{rd} step from $z = 1+\frac{e^{i\alpha}}{2}$ of length $\frac{1}{2^2}$ in the direction $e^{i2\alpha}$.

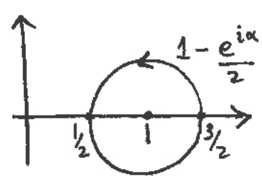
So arrive at $1+\frac{e^{i\alpha}}{2}+\frac{e^{i2\alpha}}{2^2}$.

Similarly, until in the limit arrive at $1+\frac{e^{i\alpha}}{2}+\frac{e^{2i\alpha}}{2^2}+\cdots$

We know this series equals $\dfrac{1}{1-\frac{e^{i\alpha}}{2}}$, since $\left|\frac{e^{i\alpha}}{2}\right| = \frac{1}{2} < 1$.

Now, As α varies, $\frac{e^{i\alpha}}{2}$ traces out a circle of radius $\frac{1}{2}$ around 0.

$1-\frac{e^{i\alpha}}{2}$ traces out a circle of radius $\frac{1}{2}$ centered at 1.



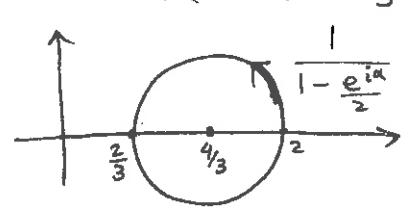
We know $\frac{1}{z}$ takes circles (not passing through 0) to circles, and fixes the real-axis.

So $\dfrac{1}{1-\frac{e^{i\alpha}}{2}}$ is a circle symmetric with respect to the x-axis.

Now $\frac{1}{2} \mapsto 2$ and $\frac{3}{2} \mapsto 2/3$ under $\frac{1}{z}$, so it is the

circle centered at $\left(2+\frac{2}{3}\right)\frac{1}{2} = \frac{4}{3}$ of radius $\frac{1}{2}\left(2-\frac{2}{3}\right) = \frac{2}{3}$.

i.e. $\left|\dfrac{1}{1-\frac{e^{i\alpha}}{2}} - \dfrac{4}{3}\right| = \dfrac{2}{3}$



$\square$

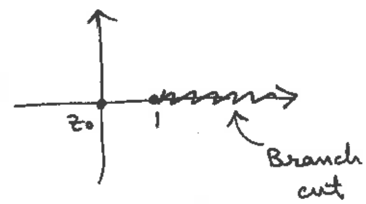
5.2.1: a) $f(z) = e^{-z}$ So $f^{(n)}(z) = (-1)^n e^{-z}$, $z_0 = 0$

$\Rightarrow f^{(n)}(0) = (-1)^n$

Thus $$f(z) = \sum_{j=0}^{\infty} \frac{f^{(j)}(0)}{j!} z^j = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} z^j = 1 - z + \frac{z^2}{2!} - \frac{z^3}{3!} + \cdots$$

for z near 0

b) $\mathrm{Log}(1-z) = f(z)$ $\qquad z_0 = 0$



$$f'(z) = -(1-z)^{-1} = (-1)(1-z)^{-1}$$

$$f''(z) = (-1)(-1)^2(1-z)^{-2} = (-1)(1-z)^{-2}$$

$$f'''(z) = (-2)(-1)(-1)^3(1-z)^{-3} = -2!(1-z)^{-3}$$

$$\vdots$$

$$f^{(n)}(z) = -(n-1)!(1-z)^{-n}$$

So $f^{(n)}(0) = -(n-1)!$ and

$$\mathrm{Log}(1-z) = \sum_{n=1}^{\infty} \frac{-(n-1)!}{n!} z^n$$ no $f^{(0)}$ term since $f(0) = 0$.

$$= \sum_{n=1}^{\infty} \frac{-(n-1)!\, z^n}{n(n-1)!} = \sum_{n=1}^{\infty} \frac{-z^n}{n}$$ for z near 0

c) $z^3 = 1 + 3(z-1) + 3(z-1)^2 + (z-1)^3$ $\qquad z_0 = 1$

Here $f(z) = z^3$ So $f'(z) = 3z^2$, $f''(z) = 6z$, $f'''(z) = 6$

and the rest are 0.

Thus $$f(z) = f(1) + f'(1)(z-1) + \frac{f''(1)}{2!}(z-1)^2 + \frac{f'''(1)}{3!}(z-1)^3 + 0$$

$$= 1 + 3(z-1) + \frac{6}{2}(z-1)^2 + \frac{6}{6}(z-1)^3$$

5.2.2: We use Thm 3 (pg. 243)

a) As e^{-z} is entire (everywhere analytic), we get that the taylor series

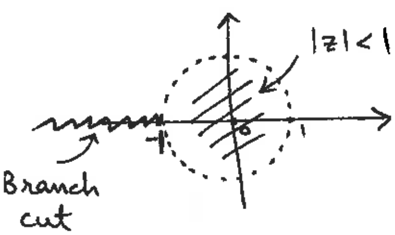
$$e^{-z} = \sum_{n=0}^{\infty} \frac{(-1)^n z^n}{n!} \quad \text{converges for all } z.$$

e) $\mathrm{Log}(1-z)$ is analytic in the disk $|z|<1$ around 0, so this is the disk on which the taylor series converges.

f) z^3 is also entire, so the series (polynomial really) converges everywhere.

5.2.4: $\alpha \in \mathbb{C}$

$$(1+z)^{\alpha} := e^{\alpha \mathrm{Log}(1+z)} = f(z)$$



$$f(0) = 1$$

$$f'(0) = e^{\alpha \mathrm{Log}(1+z)} \cdot \frac{\alpha}{1+z}\Big|_{z=0} = \alpha e^{(\alpha-1)\mathrm{Log}(1+z)}\Big|_{z=0} = \alpha (1+z)^{\alpha-1}\Big|_0 = \alpha, \quad \text{\& similarly}$$

$$f''(0) = \alpha(\alpha-1)(1+z)^{\alpha-2}\Big|_{z=0} = \alpha(\alpha-1)$$

$$f'''(0) = \alpha(\alpha-1)(\alpha-2)(1+z)^{\alpha-3}\Big|_{z=0} = \alpha(\alpha-1)(\alpha-2), \quad \ldots \text{ etc.}$$

So by the formula for taylor series and Thm 3 (pg. 243),

$$(1+z)^{\alpha} = 1 + \alpha z + \frac{\alpha(\alpha-1)z^2}{2!} + \ldots \qquad \text{for } |z|<1$$

as $\mathrm{Log}(1+z)$ is analytic on $\{|z|<1\}$,

thus so is $e^{\alpha \mathrm{Log}(1+z)} = (1+z)^{\alpha}$.

5.2.5: a) $\frac{1}{1+z} = \frac{1}{1-(-z)} = \sum_{n=0}^{\infty} (-z)^n = \sum_{n=0}^{\infty} (-1)^n z^n$ for $|-z|<1$

So $|z|<1$.

b) $e^{-z^2} = \sum_{n=0}^{\infty} \frac{(-z^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{n!}$

converges for all $-z^2$ (i.e. for all z).

c) $z^3 \sin(3z)$

$$= z^3 \left[\sum_{n=0}^{\infty} \frac{(-1)^n (3z)^{2n+1}}{(2n+1)!} \right] = z^3 \sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n+1}}{(2n+1)!} z^{2n+1}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n+1}}{(2n+1)!} z^{2n+4}$$ for all z, (since $\sin(3z)$ is entire).

5.2.14: $f(z)$ analytic in $D: |z-z_0|<R$. $f^{(k)}(z_0)=0$ for $k=0,1,\cdots$

By Thm 3 (pg. 243) $f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)(z-z_0)^k}{k!}$ for all $z \in D$.

Thus $f(z) = \sum_{k=0}^{\infty} 0 = 0$ for all $z \in D$. $\boxtimes$

5.3.2: For the series $\sum_{j=0}^{\infty} a_j (z-z_0)^j$ we have $\lim_{j\to\infty} \left| \frac{a_{j+1}}{a_j} \right| = L \geq 0$

then if $|z-z_0| < \frac{1}{L}$, we have

$$\left| \frac{a_{j+1}(z-z_0)^{j+1}}{a_j (z-z_0)^j} \right| = \left| \frac{a_{j+1}}{a_j} \right| \cdot |z-z_0|^1 < L \cdot \frac{1}{L} = 1$$

So by the Ratio Test (Thm 2 pg. 237) the series converges for this z. By the same test it diverges if $|z-z_0| > \frac{1}{L}$.

So the radius of convergence is exactly $R = \frac{1}{L}$ (∞ if $L=0$) $\boxtimes$

5.3.3: a) $f(z) = \sum_{j=0}^{\infty} j^3 z^j$

Here $\lim_{j\to\infty} \left| \frac{(j+1)^3}{j^3} \right| = 1$

So the "circle of convergence" is $|z| = \frac{1}{1} = 1$.

d) $f(z) = \sum_{k=0}^{\infty} \frac{(-1)^k k}{3^k} (z-i)^k$

As $\lim_{k\to\infty} \left| \frac{(-1)^{k+1}(k+1)}{3^{k+1}} \cdot \frac{3^k}{(-1)^k \cdot k} \right| = \lim_{k\to\infty} \left| \frac{k+1}{3k} \right| = \frac{1}{3}$,

the "circle of convergence" is $|z-i| = 3$.

e) $f(z) = \sum_{k=1}^{\infty} \frac{(3-i)^k}{k^2} (z+2)^k$.

Here $\lim_{k\to\infty} \left| \frac{(3-i)^{k+1}}{(k+1)^2} \cdot \frac{k^2}{(3-i)^k} \right| = \lim_{k\to\infty} \left| \frac{(3-i)\, k^2}{(k+1)^2} \right| = |3-i| = \sqrt{9+1} = \sqrt{10}$

So the circle of convergence is $|z+2| = \frac{1}{\sqrt{10}}$.

5.3.4: No, if there was a power series that diverged at $z = 3-i$, then it's radius of convergence (around $z_0 = 0$) would be less than $\sqrt{10}$. But $|2+3i| = \sqrt{13} > \sqrt{10}$, so it could not converge at $z = 2+3i$.

5.3.8: $f(z)$ analytic at 0 & $f(0) = f'(0) = 0$.
So f is analytic in some neighborhood $D: \{|z| < \varepsilon\}$ of 0, and on D $f(z) = 0 + 0 + \frac{f''(0)}{2!} z^2 + \frac{f'''(0)}{3!} z^3 + \cdots$

Then on D $f(z) = z^2 \left(\frac{f''(0)}{2!} + \frac{f'''(0)}{3!} z + \cdots + \frac{f^{(n)}(0)}{n!} z^{n-2} + \cdots \right)$

so define $g(z) := \sum_{n=2}^{\infty} \frac{f^{(n)}(0)}{n!} z^{n-2}$, $\underbrace{\text{analytic on } D}_{\text{as a convergent power series}}$ and $f(z) = z^2 g(z)$ $\boxtimes$