5.5.1: Find the Laurent series for $\frac{1}{z+z^2}$ in the following domains:

We note that $\frac{1}{z+z^2} = \frac{1}{z(1+z)} = \frac{1}{z} - \frac{1}{z+1}$ (by partial fractions).

a) $0<|z|<1$: Since the center is 0, we want expansion in powers of z and $\frac{1}{z}$. Note that $\frac{1}{z+1} = \sum_{n=0}^{\infty}(-z)^n$ for $|z|<1$.

So we get $\frac{1}{z+z^2} = \frac{1}{z} - \sum_{n=0}^{\infty}(-z)^n = \boxed{\frac{1}{z} - 1 + z - z^2 + \cdots}$ for $0<|z|<1$.

b) $1<|z|$: Here $\frac{1}{|z|}<1$ so we expand $\frac{1}{z+1} = \frac{1}{z}\cdot\frac{1}{1+\frac{1}{z}} = \frac{1}{z}\sum_{n=0}^{\infty}\left(-\frac{1}{z}\right)^n$,

thus obtaining $\frac{1}{z+z^2} = \frac{1}{z} - \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{-1}{z}\right)^n = \boxed{\frac{1}{z^2} - \frac{1}{z^3} + \frac{1}{z^4} - \cdots}$ for $|z|>1$.

c) $0<|z+1|<1$: Here the center is at -1, so we want an expansion in terms of powers of $z+1$ and $\frac{1}{z+1}$. So we must expand $\frac{1}{z}$ as

$1/z = \frac{-1}{1-(z+1)} = -\sum_{n=0}^{\infty}(z+1)^n$, which converges since $|1+z|<1$, so we get

$$\frac{1}{z+z^2} = -\sum_{n=0}^{\infty}(z+1)^n - \frac{1}{z+1} = \boxed{\frac{-1}{z+1} - 1 - (z+1) - (z+1)^2 - \cdots}.$$

d) $|z+1|>1$: Here expand $\frac{1}{z}$ as $\frac{1}{z} = \frac{1}{(z+1)-1} = \frac{1}{z+1}\cdot\frac{1}{1-\frac{1}{z+1}} = \frac{1}{z+1}\sum_{n=0}^{\infty}\left(\frac{1}{z+1}\right)^n$

Which converges since $\frac{1}{|z+1|}<1$, so we get:

$$\frac{1}{z^2+z} = \frac{1}{z+1}\sum_{n=0}^{\infty}\frac{1}{(z+1)^n} - \frac{1}{z+1} = \boxed{\frac{1}{(z+1)^2} + \frac{1}{(z+1)^3} + \cdots}$$ for $|z+1|>1$.

5.5.6: We know that the series for $\cos(w) = \sum_{n=0}^{\infty} \frac{(-1)^n w^{2n}}{(2n)!} = 1 - \frac{w^2}{2!} + \frac{w^4}{4!} - \cdots$

valid for all $w \in \mathbb{C}$ (i.e. finite). Thus $\cos\left(\frac{1}{3z}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\left(\frac{1}{3z}\right)^{2n}$ for $z \neq 0$.

So we get the Laurent series for $z^2 \cos\left(\frac{1}{3z}\right)$ in $|z|>0$ is,

$$z^2\cos\left(\frac{1}{3z}\right) = z^2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\left(\frac{1}{3z}\right)^{2n} = \sum_{n=0}^{\infty} \frac{(-1)^n z^2}{3^{2n}(2n)!\, z^{2n}}$$

$$= \boxed{z^2 - \frac{1}{3^2 \cdot 2!} + \frac{1}{3^4 \cdot 4!\, z^2} - \frac{1}{3^6 \cdot 6!\, z^4} + \cdots} \quad \text{for } |z|>0.$$

5.5.9: The series $\sum_{j=-\infty}^{\infty} \frac{z^j}{2^{|j|}}$ can be decomposed as:

$\sum_{j=-\infty}^{\infty} \frac{z^j}{2^{|j|}} = \sum_{j=1}^{\infty} \frac{1}{(2z)^j} + \sum_{j=0}^{\infty} \left(\frac{z}{2}\right)^j$ where the first summand converges for $\frac{1}{2|z|} < 1$

(i.e. for $|z| > \frac{1}{2}$) and diverges otherwise, and the second sum converges

when $\frac{|z|}{2} < 1$, (i.e. when $|z|<2$), and diverges otherwise.

So the annulus of convergence for $\sum_{j=-\infty}^{\infty} \frac{z^j}{2^{|j|}}$ is $\boxed{\frac{1}{2} < |z| < 2}$.

EXTRA: For f analytic in the annulus $A = \{r < |z-z_0| < R\}$, we have

$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$ where $a_n = \frac{1}{2\pi i} \int_\Gamma \frac{f(w)}{(w-z_0)^{n+1}}\, dw$, for all z in A.

If also $f(z) = \sum_{n=-\infty}^{\infty} c_n (z-z_0)^n$ $\forall z \in A$, then it's true on the

points of Γ, so $a_n = \frac{1}{2\pi i} \int_\Gamma \left(\frac{1}{(w-z_0)^{n+1}} \sum_{m=-\infty}^{\infty} c_m (w-z_0)^m\right) dw$ — eq 1

Now, the series $\sum_{n=-\infty}^{\infty} c_n (z-z_0)^n$ being a convergent Laurent series

in A, converges uniformly on closed subsets of A, in particular on the (simple) closed contour Γ. So by Thm 8 (pg. 255),

the sum of integrals converges to the integral of the sum, thus:

$$a_n = \frac{1}{2\pi i} \sum_{m=-\infty}^{\infty} C_m \cdot \int_\Gamma \frac{(\omega - z_0)^m}{(\omega - z_0)^{n+1}} \, d\omega$$ where all integrals are

0 except when $m = n$, and in that case $\int_\Gamma \frac{1}{\omega - z_0} \, d\omega = 2\pi i$

since Γ encloses z_0. Thus $a_n = \cdots + 0 + \frac{1}{2\pi i} C_n \cdot 2\pi i + 0 + \cdots = C_n$.

$\forall n \in \mathbb{Z}$. ▨

5.6.2: $f(z) = \dfrac{1}{(2\cos(z) - 2 + z^2)^2}$

Now, if we define $\frac{1}{f(z)}$ to be 0 at $z = 0$, (since $|f(z)| \to \infty$ as $z \to 0$)

we get that $\frac{1}{f(z)} = (2\cos(z) - 2 + z^2)^2$ is entire.

And $\frac{1}{f(z)} = \left(2\left(1 - \frac{z^2}{2} + \frac{z^4}{4!} - \cdots\right) - 2 + z^2\right)^2$

$$= \left(\left(\cancel{2} - \cancel{z^2} + \frac{2z^4}{4!} - \frac{2z^6}{6!} + \cdots\right) - \cancel{2} + \cancel{z^2}\right)^2$$

$$= \left(\frac{2z^4}{4!} - \frac{2z^6}{6!} + \cdots\right)^2 = (2z^4)^2 \left[\frac{1}{4!} - \frac{z^2}{6!} + \frac{z^4}{8!} - \cdots\right]^2$$

$$\Rightarrow \frac{1}{f(z)} = 4z^8 \left[\frac{1}{(4!)^2} - \left(\frac{1}{4!6!} + \frac{1}{6!4!}\right) z^2 + \left(\frac{1}{4!8!} + \frac{1}{6!6!} + \frac{1}{8!4!}\right) z^4 - \cdots\right]$$

Thus $\frac{1}{z^8 f(z)}$ is analytic and non-zero at $z = 0$, hence

$z^8 f(z)$ is analytic at $z = 0$ (and non-zero). So $f(z)$ has

a pole of order $\boxed{8}$ at $z = 0$.

5.6.3 a) $\dfrac{(z-i)^2}{(z-(2-3i))^5}$ for instance,

b) $z\, e^{\frac{1}{1-z}}$ "

c) $\dfrac{\sin(z)\, e^{\frac{1}{z-i}}}{z\,(z-1)^6}$ "

d) $\dfrac{e^{\frac{1}{z}} + e^{\frac{1}{1-z}}}{(z-(1+i))^2}$ " .

5.6.4: Pf. of Lemma 8:

($\Rightarrow$): Say f has a zero of order m at z_0. Then $f(z) = (z-z_0)^m g(z)$ where g is analytic at z_0 and $g(z_0) \neq 0$ (Thm 16 pg. 278). Since $g(z_0) \neq 0$, there is a punctured nbhd. of z_0 on which $\frac{1}{g(z)}$ is analytic. Then $\left(\frac{1}{f}\right)(z) = \frac{1}{f(z)} = \frac{1/g(z)}{(z-z_0)^m}$ on this punctured nbhd and $1/g(z_0) \neq 0$ so by Lemma 7 (pg. 281), $\frac{1}{f}$ has a pole of order m at z_0. ($\Leftarrow$): Conversely, suppose f has a pole of order m at z_0. By Lemma 7, $f(z) = \frac{g(z)}{(z-z_0)^m}$ in some punctured neighborhood of z_0 on which g is analytic and $g(z_0) \neq 0$. Then $\left(\frac{1}{f}\right)(z) = \frac{(z-z_0)^m}{g(z)}$ on a (possibly smaller) punctured nbhd. on which $\frac{1}{g}$ is analytic and $\frac{1}{g}(z_0) \neq 0$. Thus $\lim\limits_{z \to z_0} \left(\frac{1}{f}\right)(z) = 0$ exists, so define $\left(\frac{1}{f}\right)(z_0) := 0$. $\frac{1}{f}$ is then analytic at z_0 and since $\left(\frac{1}{f}\right)(z) = (z-z_0)^m \left(\frac{1}{g}\right)(z)$ where $\frac{1}{g}$ is analytic at z_0 and $\frac{1}{g}(z_0) \neq 0$, $\frac{1}{f}$ has a zero of order m at z_0. (by Thm 16 pg. 278) $\boxtimes$

5.6.6: f has a pole of order m at z_0 so in some punctured nbhd of z_0, $f(z) = \dfrac{g(z)}{(z-z_0)^m}$ where g is analytic at z_0 and $g(z_0) \neq 0$. (Lemma 7).

For z in this punctured nbhd, $f'(z) = \dfrac{(z-z_0)^m g'(z) - m(z-z_0)^{m-1} g(z)}{(z-z_0)^{2m}}$

i.e. $f'(z) = \dfrac{[(z-z_0)g'(z) - m\,g(z)]}{(z-z_0)^{m+1}} = \dfrac{h(z)}{(z-z_0)^{m+1}}$

and the function $h(z)$ is analytic at z_0 and satisfies

$h(z_0) = 0 \cdot g'(z) - m \cdot g(z_0) \neq 0$. So again by Lemma 7,

f' has a pole of order $m+1$ at z_0. $\boxed{}$

EXTRA: $1/f$ has an essential singularity at z_0 and we want to show that f does too. So suppose it didn't.

If f had a pole at z_0, then by Lemma 8 $1/f$ would have a removable singularity. That's a contradiction.

If f had a removable singularity then either $f(z_0) \neq 0$ or f has a zero at z_0. So either $1/f$ has a removable singularity or a pole at z_0. Either contradicts the assumption that $1/f$ has an ess. singularity at z_0. Thus we're forced to conclude that f must also have an ess. singularity at z_0 $\boxed{}$

5.7.1 a) e^z let $g(w) = e^{1/w}$. g has an ess. singularity at ~~z~~ $w=0$, so e^z has an essential singularity at ∞.

b) $\frac{z-1}{z+1}$, $g(w) = \frac{1/w - 1}{1/w + 1} = \frac{1-w}{1+w}$ is analytic at $w=0$,

So $\frac{z-1}{z+1}$ is analytic at $z=\infty$.

g) $f(z) = \frac{\sin(z)}{z^2}$, let $g(w) = f(1/w) = w^2 \sin(1/w)$.

$\Rightarrow g(w) = w^2\left(\frac{1}{w} - \frac{1}{3!\,w^3} + \frac{1}{5!\,w^5} - \cdots\right) = w - \frac{1}{3!\,w} - \frac{1}{5!\,w^3} + \cdots$ for $0<|w|$

has an essential singularity at $w=0$,

So $f(z)$ has an essential singularity at $z=\infty$.

h) $\frac{1}{\sin(z)}$. $g(w) = \frac{1}{\sin(1/w)}$. Now, $\sin(1/w)$ has an essential singularity at $w=0$, so $\frac{1}{\sin(1/w)}$ does as well (by extra problem above). Thus $\frac{1}{\sin(z)}$ has an essential singularity at $z=\infty$.

5.7.2: $f(z)$ is analytic at ∞. So $g(w) = f(1/w)$ is analytic at $w=0$.

So there is a positive real number $R>0$ so that

$g(w) = a_0 + a_1 w + a_2 w^2 + \cdots$ converging uniformly for $|w| \le R$.

i.e. $f(1/w) = a_0 + a_1 w + \cdots$ " "

Thus $f(z) = a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots$, converging uniformly for $\left|\frac{1}{z}\right| \le R$.

Hence $f(z) = \sum_{n=0}^{\infty} \frac{a_n}{z^n}$ converging uniformly for $\frac{1}{R} \leq |z|$

i.e. outside the disk $|z| < \frac{1}{R}$.

$\boxed{\checkmark}$

EXTRA: $P(z)$ polynomial of degree m, $Q(z)$ of degree n. $f(z) = \frac{P(z)}{Q(z)}$

Let $g(w) = f(1/w) = \frac{P(1/w)}{Q(1/w)}$.

Say $P(z) = a_0 + a_1 z + \cdots + a_m z^m$ and $a_m \neq 0 \neq b_n$

$Q(z) = b_0 + b_1 z + \cdots + b_n z^n$.

Then $g(w) = \dfrac{a_0 + a_1/w + \cdots + a_m/w^m}{b_0 + b_1/w + \cdots + b_n/w^n}$

$$= \frac{w^n}{w^m} \cdot \underbrace{\left(\frac{w^m a_0 + w^{m-1} a_1 + \cdots + a_m}{w^n b_0 + w^{n-1} b_1 + \cdots + b_n} \right)}_{\text{non-zero at } w=0, \text{ and analytic.}}$$

So at $w=0$, g has a $\begin{cases} \text{pole of order } m-n & \text{if } m>n \\ \text{zero of order } n-m & \text{if } m<n \end{cases}$

and a removable singularity otherwise (i.e. if $m=n$).

Thus by definition, $f(z) = \frac{P(z)}{Q(z)}$ has (at $z=\infty$)

either a pole of order $m-n$ if $m>n$,
or a zero of order $n-m$ if $m<n$,

or it is analytic (if $m=n$), with a value of $\frac{a_m}{b_n} = f(\infty)$.