Math 4220: Homework 7

Due Thursday, October 29 in class.

5.5. Textbook exercises: 5.5.1 (hint: partial fractions), 5.5.6, 5.5.9.

Extra problem: Let f be analytic on the annulus $A = \{r < |z - z_0| < R\}$. We know that for all $z \in A$,

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \quad \text{where} \quad a_n = \frac{1}{2\pi i} \int_\Gamma \frac{f(w)}{(w - z_0)^{n+1}} dw,$$

for any positively oriented simple closed contour Γ in A that encloses z_0. Suppose we know by some other means (as in exercises 5.5.1 and 5.5.6) that

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n \tag{1}$$

for all $z \in A$. We would like to prove that $a_n = c_n$ for all n. Prove that this is true by plugging equation (1) into the formula for a_n and interchanging the sum with the integral.

5.6. Textbook exercises: 5.6.2, 5.6.3, 5.6.4 (hint: use Lemma 7), 5.6.6.

Extra problem: Use Lemma 8 (i.e. exercise 5.6.4) to prove that if $1/f$ has an essential singularity at z_0, then f also has an essential singularity at z_0.

5.7. Textbook exercises: 5.7.1(acgh), 5.7.2.

Hint for this section: To determine the behavior of $f(z)$ at ∞, let $g(w) = f(1/w)$ and look at the Laurent series for $g(w)$ in a neighborhood of 0.

Extra problem: Let $P(z)$ be a polynomial of degree m, and let $Q(z)$ be a polynomial of degree n. Determine the behavior of $f(z) = P(z)/Q(z)$ at ∞: depending on the values of m and n, when does f have a pole or a zero at ∞, and of what order?