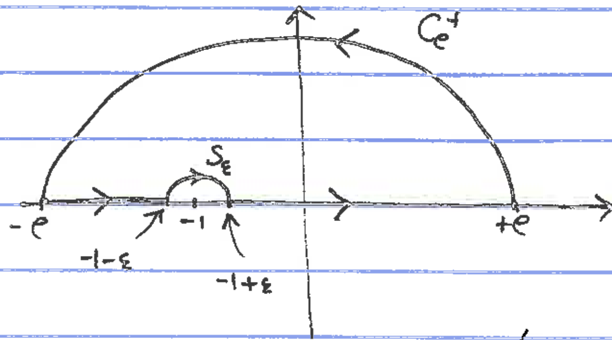
MATH 4220 HW8 SOLUTIONS

6.5.2: $\text{p.v.} \int_{-\infty}^{\infty} \frac{e^{2ix}}{x+1}\,dx = \pi i e^{-2i}$



Use contour on the right $\Gamma_{e,\varepsilon}$:

We know that $\displaystyle\int_{\Gamma_{e,\varepsilon}} \frac{e^{2iz}}{z+1}\,dz = 0$ by Cauchy Integral Thm. $\left(\begin{matrix} e>2, \\ \varepsilon<1\end{matrix}\right)$

Now, $\displaystyle\int_{C_e^+} \frac{e^{2iz}}{z+1}\,dz \longrightarrow 0$ as $e \to \infty$ by Jordan's Lemma,

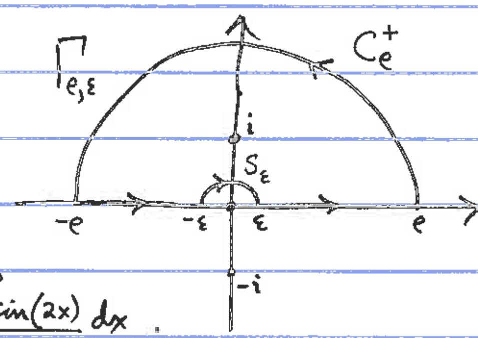
while $\displaystyle\int_{S_\varepsilon} \frac{e^{2iz}}{z+1}\,dz = -i\pi \operatorname{Res}\left(\frac{e^{2iz}}{z+1}; -1\right) = -i\pi \lim_{z\to -1}\left(e^{2iz}\right) = -i\pi e^{-2i}.$

$$\text{So } 0 = \lim_{\substack{\varepsilon\to 0^+ \\ e\to\infty}} \int_{\Gamma_{e,\varepsilon}} \frac{e^{2iz}}{z+1}\,dz$$

$$= 0 + \left(-i\pi e^{-2i}\right) + \lim_{\substack{\varepsilon\to 0^+ \\ e\to\infty}} \left(\int_{-e}^{-1-\varepsilon} \frac{e^{2ix}}{x+1}\,dx + \int_{-1+\varepsilon}^{e} \frac{e^{2ix}}{x+1}\,dx \right)$$

$$\Rightarrow i\pi e^{-2i} = \text{p.v.} \int_{-\infty}^{\infty} \frac{e^{2ix}}{x+1}\,dx \quad \boxed{}$$

6.5.4: $I = \displaystyle\int_0^{\infty} \frac{\sin(2x)}{x(x^2+1)^2}\,dx = \pi\left(\frac{1}{2} - \frac{1}{e^2}\right)$



Notice integrand is even so $2I = \text{p.v.}\displaystyle\int_{-\infty}^{\infty} \frac{\sin(2x)}{x(x^2+1)^2}\,dx$.

Moreover, it is the imaginary part of $\text{p.v.}\displaystyle\int_{-\infty}^{\infty} \frac{e^{2ix}}{x(x^2+1)^2}\,dx$.

For the latter, we use the contour above:

$$\int_{C_\rho^+} \frac{e^{2iz}}{z(z^2+1)^2}\,dz \longrightarrow 0 \quad \text{as} \quad \rho \to \infty \quad \text{by Jordan's lemma.}$$

$$\int_{S_\varepsilon} \frac{e^{2iz}}{z(z^2+1)^2}\,dz = -i\pi \operatorname{Res}\left(\frac{e^{2iz}}{z(z^2+1)^2}; 0\right) = -i\pi$$

And
$$\lim_{\substack{\rho\to\infty \\ \varepsilon\to 0^+}} \int_{\Gamma_{\rho,\varepsilon}} \frac{e^{2iz}}{z(z^2+1)^2}\,dz = \lim_{\substack{\rho\to\infty \\ \varepsilon\to 0^+}} \operatorname{Res}\left(\frac{e^{2iz}}{z(z^2+1)^2}; i\right)\cdot 2\pi i$$

Now,
$$\operatorname{Res}\left(\frac{e^{2iz}}{z(z^2+1)^2}; i\right) = \lim_{z\to i}\left(\frac{d}{dz}\left(\frac{(z-i)^2 e^{2iz}}{z(z^2+1)^2}\right)\right)$$

$$= \lim_{z\to i} \frac{d}{dz}\left(\frac{e^{2iz}}{z(z+i)^2}\right)$$

$$= \lim_{z\to i} \frac{2iz(z+i)^2 e^{2iz} - e^{2iz}\left[(z+i)^2 + 2z(z+i)\right]}{z^2(z+i)^4}$$

$$= e^{-2}\left(\frac{-2(-4) - (-4-4)}{-16}\right) = -e^{-2}$$

Thus
$$\text{p.v.}\int_{-\infty}^{\infty} \frac{e^{2ix}}{x(x^2+1)^2}\,dx - i\pi = -2\pi i e^{-2}$$

$$\Rightarrow \quad \text{p.v.}\int_{-\infty}^{\infty} \frac{e^{2ix}}{x(x^2+1)^2}\,dx = i\left(\pi - \frac{2\pi}{e^2}\right).$$

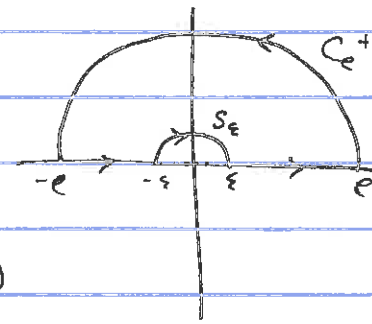
So
$$\text{p.v.}\int_{-\infty}^{\infty} \frac{\sin(2x)}{x(x^2+1)^2}\,dx = \operatorname{Im}\left(\text{p.v.}\int_{-\infty}^{\infty} \frac{e^{2ix}}{x(x^2+1)^2}\,dx\right) = \pi\left(1 - \frac{2}{e^2}\right).$$

Hence
$$\int_{0}^{\infty} \frac{\sin(2x)}{x(x^2+1)^2}\,dx = \frac{\pi}{2}\left(1 - \frac{2}{e^2}\right) \qquad \square$$

6.5.5: $\displaystyle\int_0^\infty \frac{\cos(x)-1}{x^2}\,dx = -\pi/2$

As before, note $\displaystyle\int_0^\infty \frac{\cos(x)-1}{x^2}\,dx = \frac{1}{2}\operatorname{Re}\left(\text{p.v.}\int_{-\infty}^{\infty}\frac{e^{ix}-1}{x^2}\,dx\right)$

since $\dfrac{\cos(x)-1}{x^2}$ is an even function.



We have:

$$\text{p.v.}\int_{-\infty}^{\infty}\frac{e^{ix}-1}{x^2} + \left(\int_{S_\varepsilon} + \int_{C_\rho^+}\right)\frac{e^{iz}-1}{z^2}\,dz = 0$$

simple pole!

$$\text{Here}\quad \int_{S_\varepsilon}\frac{e^{iz}-1}{z^2}\,dz = -i\pi\operatorname{Res}\left(\frac{e^{iz}-1}{z^2}, 0\right) = -i\pi\operatorname{Res}\left(\frac{iz+\frac{(iz)^2}{2!}+\cdots}{z^2}\,;0\right) = \pi$$

Jordan's lemma.

$$\text{And}\quad \int_{C_\rho^+}\frac{e^{iz}-1}{z^2}\,dz \longrightarrow 0 + \lim_{\rho\to\infty}\int_{C_\rho^+}\frac{-1}{z^2}\,dz = \lim_{\rho\to\infty}\int_0^\pi\frac{-\rho i e^{it}}{\rho^2 e^{2it}}\,dt$$

$$= \lim_{\rho\to\infty}\frac{-i}{\rho}\int_0^\pi e^{-it}\,dt = 0.$$

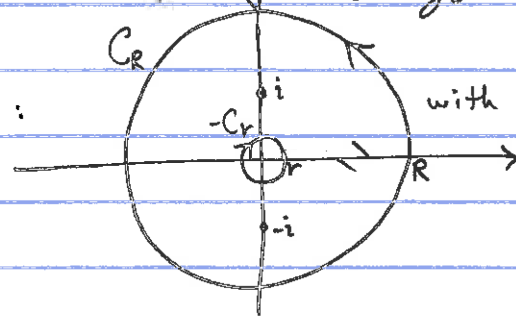
$$\text{So}\quad \int_0^\infty\frac{\cos(x)-1}{x^2}\,dx = \frac{1}{2}\operatorname{Re}\left(\text{p.v.}\int_{-\infty}^{\infty}\frac{e^{ix}-1}{x^2}\,dx\right) = \frac{1}{2}(-\pi) = \frac{-\pi}{2}.$$

∎

6.6.1: $\displaystyle\int_0^\infty\frac{\sqrt{x}}{x^2+1}\,dx = \frac{\pi}{\sqrt{2}}.$

We take $\sqrt{z} = e^{\frac{1}{2}(\operatorname{Log}|z| + i\arg_0(z))}$ (Recall $\arg_0$ is between 0 & 2π)

Consider the contour $\Gamma_{r,R}$: with $0<r<1<R$



On one hand, $\displaystyle\int_{\Gamma_{r,R}} \frac{\sqrt{z}}{z^2+1}\,dz = 2\pi i\Big(Res(f(z);i) + Res(f(z);-i)\Big)$

$$= 2\pi i\left(\frac{e^{\frac{1}{2}(\mathrm{Log}|z| + i\arg_0(z))}}{z+i}\Bigg|_{z=i} + \frac{e^{\frac{1}{2}(\mathrm{Log}|z|+i\arg_0(z))}}{z-i}\Bigg|_{z=-i}\right)$$

$$= 2\pi i\left(\frac{e^{i\pi/4}}{2i} + \frac{e^{i3\pi/4}}{-2i}\right) = \pi(\sqrt{2}).$$

On the other hand,

$$\int_{\Gamma_{r,R}} \frac{\sqrt{z}}{z^2+1}\,dz = \int_r^R \frac{\sqrt{x}}{x^2+1}\,dx + \int_R^r \frac{\sqrt{x}}{x^2+1} e^{2\pi i/2}\,dx + \left(\int_{C_R} - \int_{C_r}\right)\frac{\sqrt{z}}{z^2+1}\,dz$$

$$= \left(1-e^{i\pi}\right)\int_r^R \frac{\sqrt{x}}{x^2+1}\,dx + \left(\int_{C_R} - \int_{C_r}\right)\frac{\sqrt{z}}{z^2+1}\,dz$$

Now, $1-e^{i\pi}=2$ and for a circle of radius s,

$$\left|\int_{C_s} \frac{\sqrt{z}}{z^2+1}\,dz\right| \le \max_{C_s}\left|\frac{\sqrt{z}}{z^2+1}\right| \cdot 2\pi s \le \begin{cases} \dfrac{2\pi s\sqrt{s}}{s^2-1} & \text{if } s>1 \\[2ex] \dfrac{2\pi s\sqrt{s}}{1-s^2} & \text{if } s<1. \end{cases}$$

So $\displaystyle\lim_{R\to\infty}\int_{C_R} \frac{\sqrt{z}}{z^2+1}\,dz = 0 = \lim_{r\to 0^+}\int \frac{\sqrt{z}}{z^2+1}\,dz.$

Thus $\displaystyle\int_0^\infty \frac{\sqrt{x}}{x^2+1}\,dx = \frac{1}{2}\left(\sqrt{2}\,\pi\right) = \pi/\sqrt{2}$ ∎

6.6.2: $\displaystyle\int_0^\infty \frac{x^{\alpha-1}}{x+1}\,dx = \frac{\pi}{\sin(\pi\alpha)}$; $0<\alpha<1$.

We use the same contour and notation as above $(0<r<1<R)$

Here $z^{\alpha-1} = e^{(\alpha-1)\mathrm{Log}_0(z)} = e^{(\alpha-1)(\mathrm{Log}|z| + i\arg_0(z))}$.

Let $f(z) = \frac{z^{\alpha-1}}{z+1}$. As in the previous problem:

$$2\pi i \operatorname{Res}(f;-1) = \int_{\Gamma_{r,R}} f(z)\,dz = \left(1-e^{(\alpha-1)2\pi i}\right)\int_r^R f(x)\,dx + \left(\int_{C_R} - \int_{C_r}\right) f(z)\,dz$$

$$\Rightarrow 2\pi i \lim_{z\to -1}\left(e^{(\alpha-1)\operatorname{Log}_0(z)}\right) = \left(1-e^{2\pi i\alpha}\right)\int_r^R f(x)\,dx + \left(\int_{C_R} - \int_{C_r}\right) f(z)\,dz.$$

$$\Rightarrow -2\pi i\left(e^{i\alpha\pi}\right) = \quad '' \quad\quad ''$$

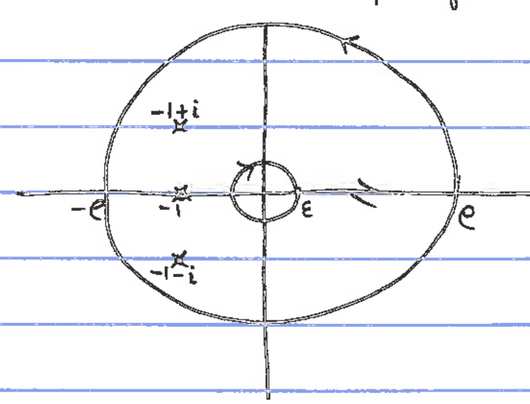
Now, $\left|\int_{C_R} \frac{z^{\alpha-1}}{z+1}\,dz\right| \leq \max_{C_R}\left|\frac{z^{\alpha-1}}{z+1}\right| \cdot 2\pi R \leq \frac{2\pi R}{(R-1)R^{1-\alpha}} \longrightarrow 0$ as $R\to\infty$ since $0<\alpha<1$

And, $\displaystyle\int_{C_r} \frac{z^{\alpha-1}}{z+1}\,dz = \int_0^{2\pi} \frac{(re^{i\theta})^{\alpha-1}}{re^{i\theta}+1}\, ri e^{i\theta}\,d\theta$

$$= r^{\alpha}\int_0^{2\pi} \frac{e^{i\theta(\alpha-1)}}{\underbrace{re^{i\theta}+1}_{\text{bounded for } r \text{ small.}}}\, i\,d\theta \longrightarrow 0 \quad \text{as } r\to 0. \quad \text{since } 0<\alpha<1.$$

Thus $\displaystyle\int_0^\infty f(x)\,dx = \frac{-2\pi i e^{i\alpha\pi}}{1-e^{2\pi i\alpha}} = +\pi\cdot\left(\frac{2i}{e^{\pi i\alpha}-e^{-\pi i\alpha}}\right) = \frac{+\pi}{\sin(\pi\alpha)}$ ☒

6.6.9 a): $\displaystyle\int_0^\infty \frac{x}{(x+1)(x^2+2x+2)}\,dx$ We follow the notation and technique of example 4.



Let $f(z) = \dfrac{\operatorname{Log}_0(z)\cdot z}{(z+1)(z^2+2z+2)}$

Note that $\int_{\varepsilon}^{e} \frac{(\operatorname{Log} x)\, x}{(x+1)(x^2+2x+2)}\,dx + \int_{e}^{\varepsilon} \frac{(\operatorname{Log} x + 2\pi i)\, x}{(x+1)(x^2+2x+2)}\,dx = -2\pi i \int_{\varepsilon}^{e} f(x)\,dx.$

While $\int_{\Gamma_\varepsilon} f(z)\,dz \to 0$ by the same reason,

and $\left| \int_{C_e} f(z)\,dz \right| \leq \frac{\sqrt{(\log e)^2+(2\pi)^2}\; 2\pi e^2}{(e-1)(e^2-2e-2)} = \frac{2\pi}{e} \frac{\sqrt{(\log e)^2+(2\pi)^2}}{(e^2-2e-2)} \to 0$

as $e \to \infty$ since $\frac{\log e}{e} \to 0$

Thus $\int_{I_{\varepsilon,e}} f(z)\,dz \longrightarrow -2\pi i \int_0^\infty \underset{\operatorname{Log}_0(x)}{f(x)}\,dx = -2\pi i \int_0^\infty \frac{x\,dx}{(x+1)(x^2+2x+2)}$

On the other hand, using residues,

$$\int_{I_{\varepsilon,e}} f(z)\,dz = 2\pi i\left(\operatorname{Res}(f;-1) + \operatorname{Res}(f;-1+i) + \operatorname{Res}(f;-1-i)\right)$$

$$= 2\pi i\left(\frac{(-1).(i\pi)}{-i^2} + \frac{(-1+i)(\log\sqrt{2} + i\,3\pi/4)}{2i^2} + \frac{(-1-i)(\log\sqrt{2} + i\,5\pi/4)}{-i(-2i)} \right)$$

$$= 2\pi i\left(-\cancel{i\pi} + \log\sqrt{2} + \cancel{i\pi} - \pi/4 \right)$$

So $\int_0^\infty \frac{x}{(x+1)(x^2+2x+2)}\,dx = \frac{2\pi i\left(\frac{1}{2}\log 2 - \pi/4\right)}{-2\pi i} = \frac{\pi}{4} - \frac{1}{2}\operatorname{Log} 2$

6.6.9 b) $\int_0^\infty \frac{1}{x^3+1}\,dx$ Let $f(z) = \frac{\operatorname{Log}_0(z)}{z^3+1}$

As above $\int_{I_{\varepsilon,e}} f(z)\,dz \to -2\pi i \int_0^\infty \frac{1}{x^3+1}\,dx$

since $\left| \int_{\Gamma_\varepsilon} f(z)\,dz \right| \leq \frac{\sqrt{(\log \varepsilon)^2+(2\pi)^2}\,.\,2\pi\varepsilon}{1-\varepsilon^3} \to 0$ as $\varepsilon \to 0^+$

And $\left|\int_{C_\rho} f(z)\,dz\right| \leq \dfrac{\sqrt{(\log\rho)^2+(2\pi)^2}\;2\pi\rho}{\rho^3-1} \longrightarrow 0$ as $\rho\to\infty$.

On the other hand $\displaystyle\int_{I_{\varepsilon,\rho}} f(z)\,dz = 2\pi i\left(\operatorname{Res}(f;-1)+\operatorname{Res}(f;e^{i\pi/3})+\operatorname{Res}(f;e^{i5\pi/3})\right)$

Since $f(z) = \dfrac{L_0(z)}{(z+1)(z-e^{i\pi/3})(z-e^{i5\pi/3})}$, we get

$$\int_{I_{\varepsilon,\rho}} f(z)\,dz = 2\pi i\left(\frac{i\pi}{(-1-e^{i\pi/3})(-1-e^{i5\pi/3})} + \frac{i\pi/3}{(e^{i\pi/3}+1)(e^{i\pi/3}-e^{i5\pi/3})} + \frac{i5\pi/3}{(e^{i5\pi/3}+1)(e^{i5\pi/3}-e^{i\pi/3})}\right)$$

So $\displaystyle\int_0^\infty \frac{1}{x^3+1}\,dx = \frac{1}{-2\pi i}\left(2\pi i\int_{I_{\varepsilon,\rho}} f(z)\,dz\right) = -\int_{I_{\varepsilon,\rho}} f(z)\,dz$

$$\Rightarrow \quad = -i\pi\left(\frac{1}{(-3)} + \frac{1/3}{2\left(\frac{3}{2}+\frac{\sqrt{3}}{2}i\right)\sqrt{3}i} + \frac{5/3}{2\left(\frac{3}{2}-\frac{\sqrt{3}}{2}i\right)(-\sqrt{3}i)}\right)$$

$$= \frac{\sqrt{3}}{9}\cdot 2\pi$$

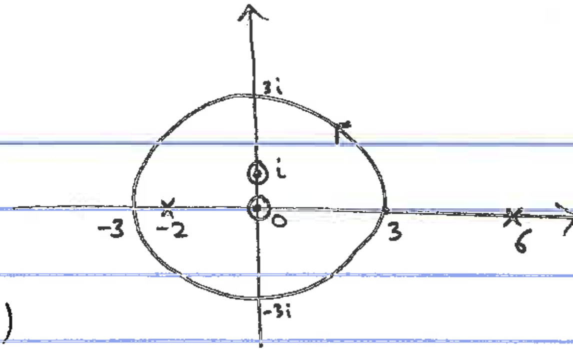
(using $e^{i\pi/3} = \frac{1}{2}+\frac{\sqrt{3}}{2}i$ and $e^{i5\pi/3} = \frac{1}{2}-\frac{\sqrt{3}}{2}i$).

6.7.2: $P(z) = a_n z^n + \cdots + a_0$ $\quad a_n \neq 0$. By the Argument Principle

$$\oint_{|z|=R} \frac{P'(z)\,dz}{P(z)} = 2\pi i\,(\#\text{ of zeroes of } P - \#\text{ of poles of } P).$$

For sufficiently large R, all n roots of P are inside $|z|=R$. And there are no poles so the result follows.

6.7.3: $f(z) = \dfrac{z^2(z-i)^3 e^z}{3(z+2)^4(3z-18)^5}$



$$\frac{1}{2\pi i}\oint_{|z|=3} \frac{f'(z)}{f(z)}\,dz = N_0(f) - N_p(f)$$

$$= (3+2) - (4) = 1$$

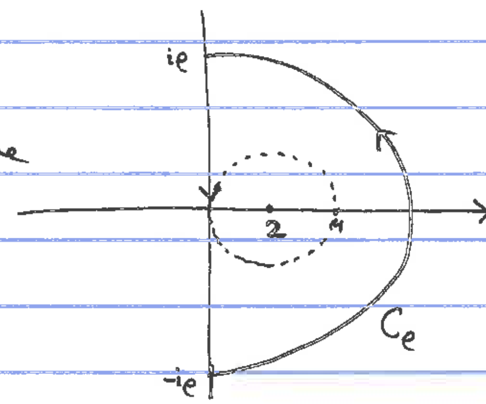
6.7.6: On the unit circle $|z|=1$, $\underset{h}{|z^6|} = 1 < 3 = 4|z|^2 - 1 \le \underset{f}{|4z^2-1|}$

So $4z^2-1$ and z^6+4z^2-1 must have the same number of zeroes in the unit disk $|z|<1$ (Rouché)

$4z^2-1$ has two roots $z = \pm 1/2$.

Thus z^6+4z^2-1 also has two roots in the unit disk.

6.7.10: Let $f(z) = z-2$ and $h(z) = e^{-z}$ and let C_ρ be the contour on the right, with $\rho > 4$.



On C_ρ, $|f(z)| \geq 2$

and $|h(z)| = |e^{-z}| = e^{-x} \le 1$ where $z = x+iy$.

as $x \geq 0$ in the right half plane.

So $|f(z)| > |h(z)|$ on C_ρ and thus f and $f+h$ have the same number of zeroes in C_ρ (for all $\rho \gg 4$)

Since f has only one zero (at 2),

$f+h = z-2+e^{-z}$ also has only one zero inside C_ρ.

Since this holds for all $e > 4$, it holds in the entire right half plane. Thus $z = 2 - e^{-z}$ has only one solution in the right half plane.

Say z_0 is this solution, i.e. $z_0 = 2 - e^{-z_0}$.

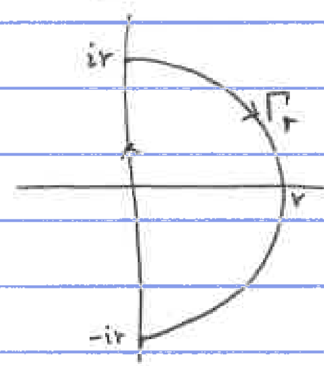
Taking conjugates on both sides we get

$$\overline{z_0} = \overline{2 - e^{-z_0}} = 2 - \overline{e^{-z_0}} = 2 - e^{-\overline{z_0}}$$

Thus $\overline{z_0}$ is also a solution, (and also in the right half plane!)

So if z_0 was not real, we would have two solutions ↯.

So $z_0 = \overline{z_0}$, thus z_0 is real. ☑



6.7.21 $F(z) = 1 + P(z)$.

Note that $\frac{1}{2\pi i}\int_{\Gamma_r} \frac{P'(z)}{1+P(z)}\,dz = \frac{1}{2\pi i}\int_{P(\Gamma_r)} \frac{dw}{1+w} = m$

by substitution $w = P(z)$, since $P(\Gamma_r)$ goes around $w=-1$ 'm' times.

But $\frac{1}{2\pi i}\int_{\Gamma_r} \frac{P'(z)}{1+P(z)}\,dz = \frac{1}{2\pi i}\int_{\Gamma_r} \frac{F'(z)}{F(z)}\,dz = -(N_0(F) - N_p(F))$

As Γ_r is negatively oriented.

When r is large enough, we have $N_p(F) = N_p(P) = n = m$, so

$m = m - N_0(F)$, hence $N_0(F) = 0$ inside Γ_r.

Thus F has no zeroes in the right half plane! ☑