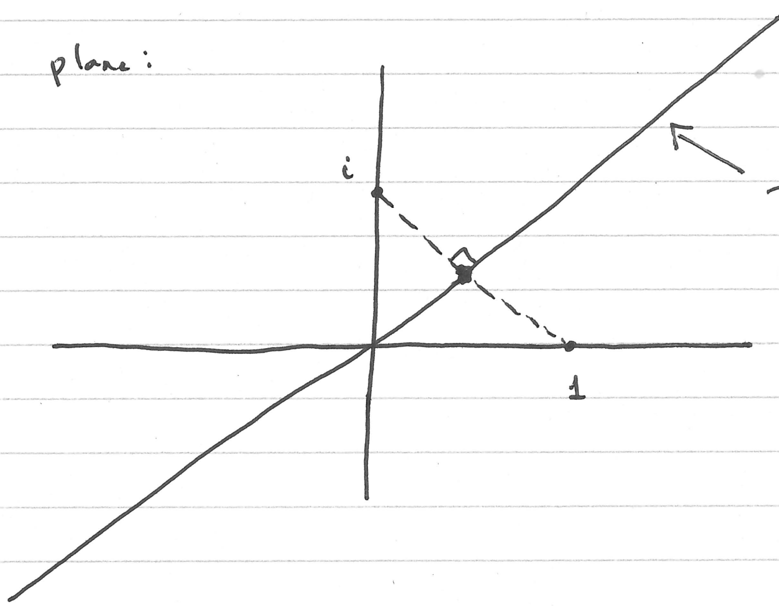
Math 4220 : Prelim 1 Practice Exam Solutions

1. Solve : $|z - 1| = |z - i|$.

Method 1: The distance from z to 1 is the same as the distance from z to i. Therefore, z lies on the perpendicular bisector of the line segment connecting 1 and i in the complex plane:



This line is the set of solutions.

Answer:
$z = x + iy$ where $y = x$.

Method 2: $|z - 1|^2 = |z - i|^2$,

$$z - 1 = (x-1) + iy, \qquad z - i = x + i(y-1)$$

Thus,
$$(x-1)^2 + y^2 = x^2 + (y-1)^2$$
$$x^2 - 2x + 1 + y^2 = x^2 + y^2 - 2y + 1$$
$$-2x = -2y, \quad \text{so} \quad x = y.$$

2. $e^{e^{1+i}} = ?$

First, $e^{1+i} = e^1 \cdot e^{i \cdot 1} = e\left(\cos 1 + i \sin 1\right)$

$$\Rightarrow e^{e^{1+i}} = e^{e\cos 1 + i(e \sin 1)}$$

$$= e^{e\cos 1} e^{i(e\sin 1)}$$

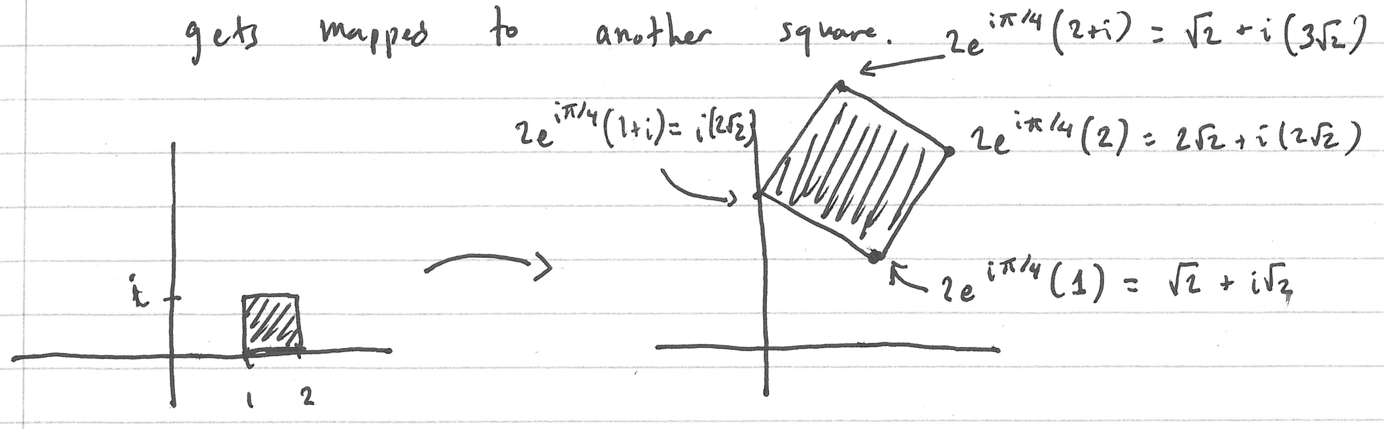
$$= e^{e\cos 1}\left[\cos(e\sin 1) + i\sin(e\sin 1)\right]$$

$$= e^{e\cos 1}\cos(e\sin 1) + i\left[e^{e\cos 1}\sin(e\sin 1)\right].$$

3. The map $z \mapsto 2e^{i\pi/4} z$ <u>rotates</u> by an angle of $\pi/4$ and then <u>scales</u> by a factor of 2. Therefore the square

$$\{z \in \mathbb{C} : 1 \le \mathrm{Re}(z) \le 2,\ 0 \le \mathrm{Im}(z) \le 1\}$$

gets mapped to another square.



4. $$\lim_{z\to 0} \frac{e^{3z} - 1}{z} = \lim_{z \to 0} \frac{e^{3z} - e^{3\cdot 0}}{z - 0} = f'(0)$$

where $f(z) = e^{3z}$. Therefore, the limit is $3e^{3z}$ evaluated at $z = 0$, which is 3.

Alternatively: $$e^{3z} = 1 + 3z + \frac{(3z)^2}{2!} + \cdots$$

$$e^{3z} - 1 = 3z + \frac{(3z)^2}{2!} + \cdots$$

$$\frac{e^{3z} - 1}{z} = 3 + \frac{9z}{2!} + \text{(higher powers of } z)$$

So, the limit as $z \to 0$ is 3.

5. Where is $f(x+iy) = (2x^2 - x - y) + i\,(2xy - x + y)$ differentiable? Answer: wherever the Cauchy-Riemann equations are satisfied.

$u(x,y) = 2x^2 - x - y$, $\quad v(x,y) = 2xy - x + y$

$$\frac{\partial u}{\partial x} = 4x - 1, \quad \frac{\partial u}{\partial y} = -1, \quad \frac{\partial v}{\partial x} = 2y - 1, \quad \frac{\partial v}{\partial y} = 2x + 1$$

So, $4x - 1 = 2x + 1$ and $2y - 1 = 1$

$\Rightarrow$ $x = 1$, $y = 1$. Thus f is differentiable at $1 + i$ and nowhere else.

6. $u(x,y) = x^3 - 3xy^2 - 2x$

Harmonic: $\frac{\partial u}{\partial x} = 3x^2 - 3y^2 - 2$, $\frac{\partial^2 u}{\partial x^2} = 6x$

$\frac{\partial u}{\partial y} = -6xy$, $\frac{\partial^2 u}{\partial y^2} = -6x$

$\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$ ✓

Find harmonic conjugate v such that $v(1,1) = 4$:

By Cauchy-Riemann equations,

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 3x^2 - 3y^2 - 2 \Rightarrow v(x,y) = 3x^2y - y^3 - 2y + C(x)$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 6xy \Rightarrow v(x,y) = 3x^2y + D(y)$$

So $v(x,y) = 3x^2y - y^3 - 2y + c$.

Plug in $(1,1)$: $v(1,1) = c = 4$, so

$$v(x,y) = 3x^2y - y^3 - 2y + 4.$$

7. Given: $f = u + iv$ is analytic. So

$f^2 = (u+iv)^2 = (u^2 - v^2) + i(2uv)$, meaning that

both $u^2 - v^2$ and $2uv$ are harmonic. It follows immediately that uv is harmonic. To find a harmonic conjugate of uv, we manipulate $f^2 = (u^2 - v^2) + i(2uv)$ into a new function whose real part is uv. The way to do this is multiplication by $-\frac{1}{2}i$:

$$-\frac{1}{2} i f^2 = -\frac{1}{2} i (u^2 - v^2) + \frac{1}{2}(2uv) = uv + i\left(\frac{v^2 - u^2}{2}\right).$$

Since $g(z) = -\frac{1}{2} i f^2(z)$ is analytic, $\frac{v^2 - u^2}{2}$ is a harmonic conjugate of uv.

8. Partial fraction decomposition of $\dfrac{z^3 - 2z^2 + 2}{(z-2)^4}$:

$$\frac{z^3 - 2z^2 + 2}{(z-2)^4} = \frac{A}{z-2} + \frac{B}{(z-2)^2} + \frac{C}{(z-2)^3} + \frac{D}{(z-2)^4},$$

$$z^3 - 2z^2 + 2 = A(z-2)^3 + B(z-2)^2 + C(z-2) + D.$$

Plug in $z = 2$: $2 = D$.

Differentiate: $3z^2 - 4z = 3A(z-2)^2 + 2B(z-2) + C$.

Plug in $z = 2$: $4 = C$.

Differentiate : $6z - 4 = 6A(z-2) + 2B$.

Plug in $z = 2$: $8 = 2B \Rightarrow B = 4$.

Differentiate: $6 = 6A \Rightarrow A = 1.$

Thus, $\dfrac{z^3 - z^2 + 2}{(z-2)^4} = \dfrac{1}{z-2} + \dfrac{4}{(z-2)^2} + \dfrac{4}{(z-2)^3} + \dfrac{2}{(z-2)^4}.$

9. $\sin^2(z) + \cos^2(z) = 1$?

$\sin(z) = \dfrac{e^{iz} - e^{-iz}}{2i}$, $\cos(z) = \dfrac{e^{iz} + e^{-iz}}{2}$.

$\sin^2(z) = -\frac{1}{4}\left[e^{2iz} - 2 + e^{-2iz}\right]$, $\cos^2(z) = \frac{1}{4}\left[e^{2iz} + 2 + e^{-2iz}\right]$

$\Rightarrow \sin^2(z) + \cos^2(z) = \frac{1}{4}(2+2) = 1.$ ✓

10. $F(z) = \mathrm{Log}\left(\dfrac{z-3i}{z}\right)$, $G(z) = \mathrm{Log}(z-3i) - \mathrm{Log}(z)$.

F is analytic except at $z=0$ and where $\dfrac{z-3i}{z} \in (-\infty, 0]$.

Let $c = \dfrac{z-3i}{z}$, then $cz = z - 3i$, $(1-c)z = 3i$,

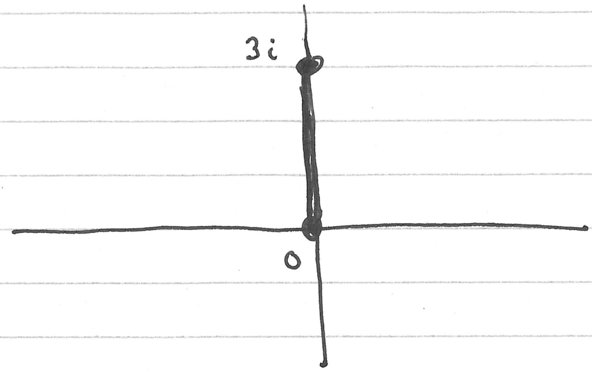
$z = \left(\dfrac{3}{1-c}\right)i$. When $-\infty < c \leq 0$, that means

$1 \leq 1-c < \infty$, so $0 < \frac{3}{1-c} \leq 3$. The

values of z at which F is not analytic are

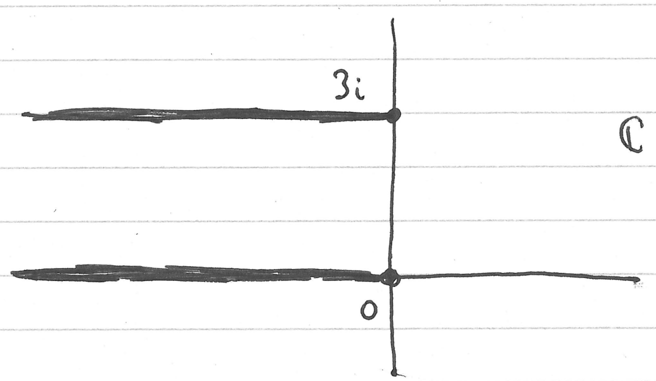
therefore $z = 0$ and $z = ki$ where $0 < k \leq 3$.

Graph on next page:



F is analytic on $\mathbb{C} \setminus \{ki : 0 \le k \le 3\}$.

For G, we need both $\mathrm{Log}(z-3i)$ and $\mathrm{Log}(z)$ to be analytic. $\mathrm{Log}(z)$ is analytic except when $z \in (-\infty, 0]$. $\mathrm{Log}(z-3i)$ is analytic except when $z - 3i \in (-\infty, 0]$. Graph:



G is analytic on $\mathbb{C} \setminus \left(\{x + 3i : x \le 0\} \cup (-\infty, 0] \right)$.

Note that F and G have the same branch points, which is predictable since both are branches of the same multiple-valued function.

11. $(1+i)^{2\pi i} = e^{(2\pi i)\log(1+i)}$. We compute:

$$\log(1+i) = \ln|1+i| + i\arg(1+i)$$

$$= \ln\sqrt{2} + i\left(\frac{\pi}{4} + 2k\pi\right)$$

$$= \frac{1}{2}\ln 2 + i\left(\frac{\pi}{4} + 2k\pi\right).$$

Therefore, $e^{(2+i)\log(1+i)} = e^{(2+i)\left(\frac{1}{2}\ln 2 + i\left[\frac{\pi}{4}+2k\pi\right]\right)}$

$$= e^{\left(\ln 2 - \left[\frac{\pi}{4}+2k\pi\right]\right) + i\left(\frac{1}{2}\ln 2 + \frac{\pi}{2} + 4k\pi\right)}.$$

First part: $e^{\ln 2 - \frac{\pi}{4} - 2k\pi} = 2e^{-\frac{\pi}{4}}e^{-2k\pi}$.

Second part: $e^{i\left(\frac{1}{2}\ln 2 + \frac{\pi}{2} + 4k\pi\right)} = e^{i\left(\frac{1}{2}\ln 2\right)} e^{i\frac{\pi}{2}} e^{i(4k\pi)}$.

We know $e^{i(4k\pi)} = 1$, so this becomes

$$e^{i\left(\frac{\pi}{2} + \frac{1}{2}\ln 2\right)}.$$

Conclusion: $(1+i)^{2+i} = 2e^{-\frac{\pi}{4}}e^{-2k\pi} \cdot e^{i\left(\frac{\pi}{2}+\frac{1}{2}\ln 2\right)}$

which is infinitely many points that all have argument $\frac{\pi}{2} + \frac{1}{2}\ln 2$ (in the second quadrant).