Math 4220: Prelim 1 Practice Exam

This practice exam is much longer than the actual prelim!

1. Which complex numbers $z = x + iy$ satisfy $|z - 1| = |z - i|$? Draw the solutions on a graph.

2. Write $e^{e^{1+i}}$ in the form $a + bi$.

3. Draw on a graph the image of the square $\{z \in \mathbf{C} : 1 \leq \mathrm{Re}(z) \leq 2, 0 \leq \mathrm{Im}(z) \leq 1\}$ under the map $z \mapsto 2e^{i\pi/4}z$.

4. Compute $\lim_{z\to 0} \dfrac{e^{3z} - 1}{z}$.

5. Let $f(x + iy) = (2x^2 - x - y) + i(2xy - x + y)$. Find all locations $x + iy$ at which f is differentiable.

6. Show that $u(x, y) = x^3 - 3xy^2 - 2x$ is harmonic, and find a harmonic conjugate $v(x, y)$ such that $v(1, 1) = 4$.

7. Suppose that u is a harmonic function and v is a harmonic conjugate of u. Prove that the function uv is harmonic, and find a harmonic conjugate of uv. *Hint:* Let $f = u + iv$. What is f^2?

8. Find the partial fraction decomposition of $\dfrac{z^3 - 2z^2 + 2}{(z - 2)^4}$.

9. Using the definitions of $\sin(z)$ and $\cos(z)$ in terms of the complex exponential function, prove that $\sin^2(z) + \cos^2(z) = 1$.

10. Let $F(z) = \mathrm{Log}\left(\dfrac{z - 3i}{z}\right)$ and $G(z) = \mathrm{Log}(z - 3i) - \mathrm{Log}(z)$. On separate graphs, draw the domains of analyticity for F and G. (Note that both F and G are branches of the multiple-valued function $\log\left(\dfrac{z - 3i}{z}\right)$.)

11. Find and graph all values of $(1 + i)^{2+i}$.