Math 4220: Prelim 1

September 24, 2015

Name:

Instructions: This exam is 75 minutes long. It has 8 pages including the cover and 6 questions worth a total of 100 points. No written or electronic aids are allowed.

Please fully explain all your answers. If you need more space to answer a question, use the back of the *preceding* sheet or the blank sheet at the end of the exam. Label your work clearly if you use extra space.

Academic integrity is expected of all Cornell University students at all times, whether in the presence or absence of members of the faculty. Understanding this, I declare I shall not give, use, or receive unauthorized aid in this examination.

Please sign below to indicate that you have read and agree to these instructions.

Signature:

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Total score:	

1. (15 points) Find all solutions $z = re^{i\theta} \in \mathbf{C}$ to the equation $z^2 = \overline{z}$.

$$z = re^{i\theta}, \quad z^2 = r^2 e^{i(2\theta)}$$

$$\overline{z} = re^{-i\theta}$$

so $r^2 = r$ and $2\theta = -\theta + 2k\pi$

$$\Downarrow \qquad\qquad \Downarrow$$

$r(r-1) = 0$ $\qquad 3\theta = 2k\pi$

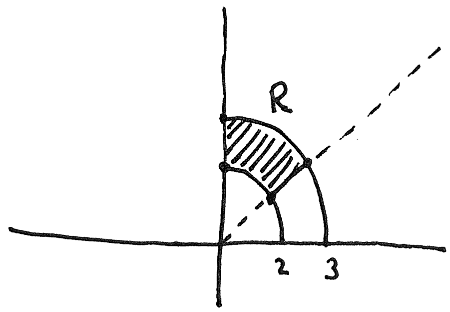
$r = 0$ or 1 $\qquad \theta = \frac{2k\pi}{3}$

$\theta = 0, \frac{2\pi}{3}, \frac{4\pi}{3}$ (then it repeats)

Solutions: $z = 0, 1, e^{i(\frac{2\pi}{3})}, e^{i(\frac{4\pi}{3})}$.

2. (15 points) (a) Draw the following region in the complex plane:

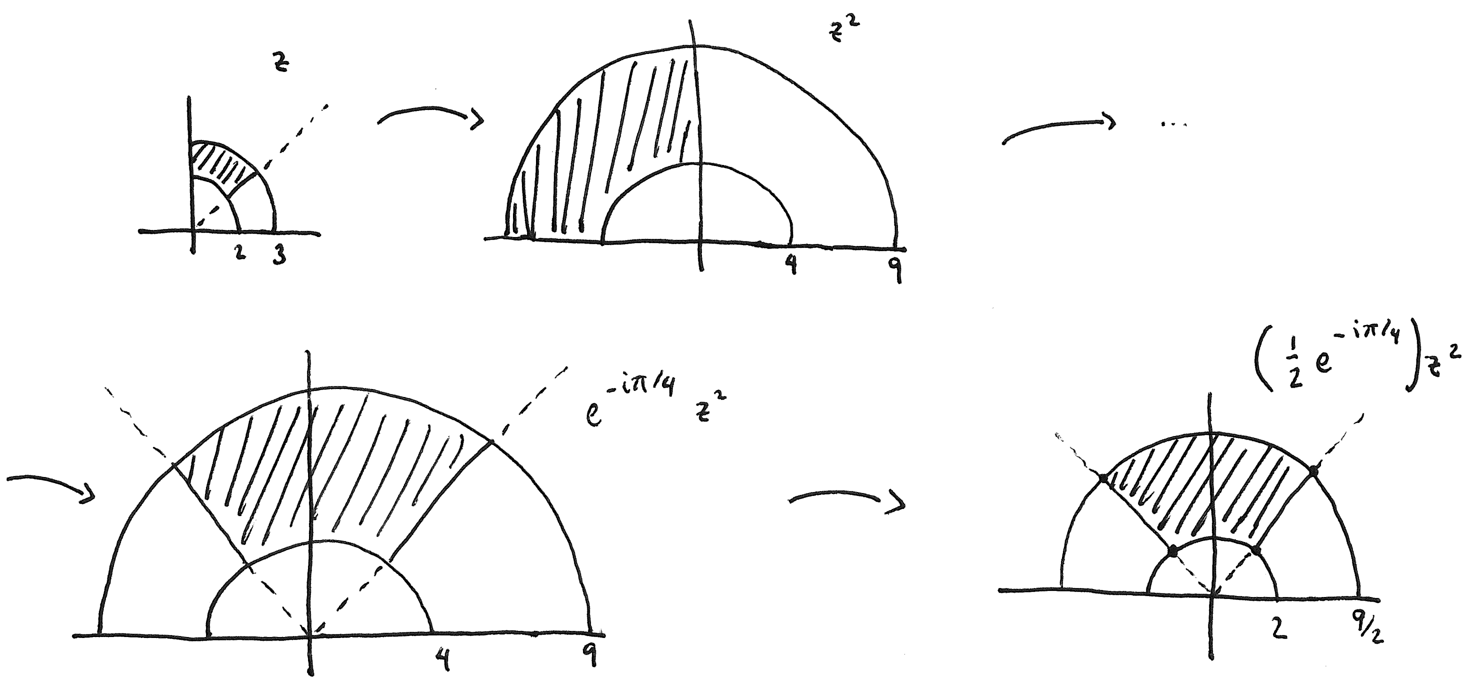
$$R = \left\{ z \in \mathbf{C} : 2 \leq |z| \leq 3, \frac{\pi}{4} \leq \mathrm{Arg}(z) \leq \frac{\pi}{2} \right\}$$



Corners of R:

$2e^{i\pi/4}$, $3e^{i\pi/4}$, $2i$, $3i$

(b) On another graph, draw the image of R under the map $z \mapsto \left(\frac{1}{2}e^{-i\pi/4}\right) z^2$. Label the four corners of the new region either in $a+bi$ or $re^{i\theta}$ form, whichever you prefer.



Corners: $2e^{i\pi/4}$, $2e^{i(3\pi/4)}$,

$\frac{9}{2}e^{i\pi/4}$, $\frac{9}{2}e^{i(3\pi/4)}$

3. (20 points) Let $f(x+iy) = (x^2 - y^2 - 3x) + i(x^2 - y^2 + 2xy + y)$.

(a) Verify that both the real and imaginary parts of f are harmonic functions.

$u(x,y) = x^2 - y^2 - 3x$, $v(x,y) = x^2 - y^2 + 2xy + y$

$\frac{\partial u}{\partial x} = 2x - 3$, $\frac{\partial u}{\partial y} = -2y$, $\frac{\partial v}{\partial x} = 2x + 2y$, $\frac{\partial v}{\partial y} = -2y + 2x + 1$

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2 - 2 = 0$ ✓ so, u is harmonic.

$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 2 - 2 = 0$ ✓ so, v is harmonic.

(b) Find all $z \in \mathbf{C}$ at which f is differentiable (in the complex sense).

Cauchy - Riemann equations:

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$,

$$2x - 3 = -2y + 2x + 1$$
$$-3 = -2y + 1$$
$$2y = 4$$
$$y = 2$$

$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$,

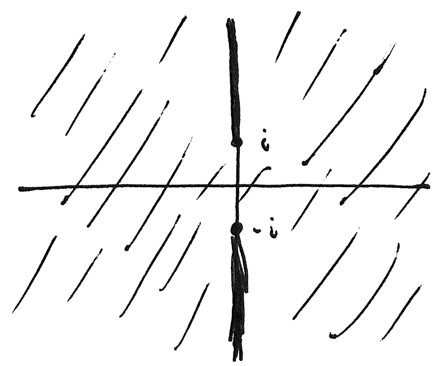
$$-2y = -(2x + 2y)$$
$$0 = 2x$$
$$x = 0$$

The only $z \in \mathbb{C}$ at which f is differentiable is $z = 0 + 2i$ $= 2i$.

4. (20 points) The principal branch of the multiple-valued function $(z^2+1)^{1/3}$ is $e^{(1/3)\,\mathrm{Log}(z^2+1)}$.

(a) On a graph, draw the domain of analyticity for this branch.

$e^{\frac{1}{3}\mathrm{Log}(z^2+1)}$ is analytic except where $z^2+1 \in (-\infty, 0]$. Equivalent to $z^2 \in (-\infty, -1]$. If z^2 is a negative real number, z is purely imaginary; the condition $z^2 \leq -1$ means that $|z| \geq 1$, so the points at which the function fails to be analytic are values $z = bi$ with $|b| \geq 1$. Domain of analyticity:



(b) Suppose $z \in \mathbf{R}$. Prove that $e^{(1/3)\,\mathrm{Log}(z^2+1)} = \sqrt[3]{z^2+1}$, the usual real-valued cube root of z^2+1.

If $z \in \mathbb{R}$ then $z^2+1 \geq 1$ is a positive real number. Therefore, $\mathrm{Log}(z^2+1) = \ln|z^2+1| + i \cdot 0 = \ln(z^2+1)$,

and
$$e^{\frac{1}{3}\mathrm{Log}(z^2+1)} = e^{\frac{1}{3}\ln(z^2+1)} = \left(e^{\ln(z^2+1)}\right)^{\frac{1}{3}}$$
$$= \sqrt[3]{z^2+1}.$$

5. (15 points) Prove that $\sin(z) = \cos\left(\frac{\pi}{2} - z\right)$ for all $z \in \mathbf{C}$.

$$\cos\left(\frac{\pi}{2} - z\right) = \frac{e^{i\left(\frac{\pi}{2} - z\right)} + e^{-i\left(\frac{\pi}{2} - z\right)}}{2}$$

$$= \frac{e^{i\pi/2}\, e^{-iz} + e^{-i\pi/2}\, e^{iz}}{2}$$

$$= \frac{i\, e^{-iz} + (-i)\, e^{iz}}{2}$$

(multiply top and bottom by i)

$$= \frac{-e^{-iz} + e^{iz}}{2i} = \sin(z).$$

6. (15 points) Let f be an analytic function. Suppose that $f(z_0) = re^{i\theta}$ and $f'(z_0) = se^{i\phi}$, with both $r, s > 0$. Imagine that you are standing at z_0 and start walking in the direction of angle ψ, so that your position after a short period of time is $z = z_0 + te^{i\psi}$ where t is a small positive number. As you walk, you keep track of how the value $f(z)$ is changing. Which angle ψ should you choose in order for the magnitude $|f(z)|$ to increase as quickly as possible? Your answer should be a formula for ψ in terms of z_0, r, s, θ, ϕ. (Hint: Not all five of those variables will appear in the formula.)

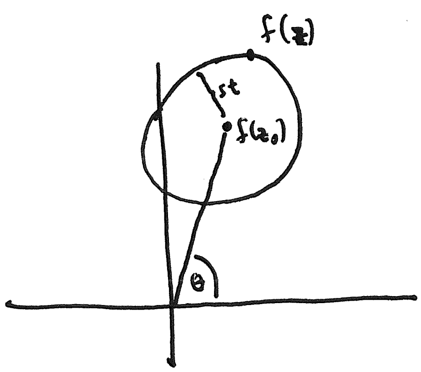
Near z_0, $f(z) \approx f(z_0) + f'(z_0)(z - z_0)$

$$= re^{i\theta} + se^{i\phi}(z - z_0).$$

If $z - z_0 = te^{i\psi}$, $f(z) \approx re^{i\theta} + se^{i\phi}(te^{i\psi})$

$$= re^{i\theta} + ste^{i(\phi+\psi)}.$$

Therefore, after having walked a distance of t away from z_0, the value of $f(z)$ lies on the circle of radius st centered at $f(z_0) = re^{i\theta}$:



To maximize $|f(z)|$ we want to be moving directly away from the origin, which happens when the angle $\phi + \psi$ matches the angle θ. Thus we should choose $\psi = \theta - \phi$.