Math 4220: Prelim 2 Practice Exam Solutions

1. (a) Let the contour Γ go from $z = 2$ to $z = -2$ counterclockwise around the upper semicircle $\{|z| = 2,\ \mathrm{Im}(z) \geq 0\}$. By parametrizing Γ, compute

$$\int_\Gamma \frac{1}{z} dz.$$

Solution: Parametrize Γ by $z(t) = 2e^{it}$, $0 \leq t \leq \pi$. Then

$$\int_\Gamma \frac{1}{z} dz = \int_0^\pi \frac{1}{2e^{it}} 2ie^{it} dt = \int_0^\pi i dt = i\pi.$$

(b) Let $L(z)$ be the branch of $\log(z)$ that takes the argument to be in the interval $(-\pi/2, 3\pi/2]$. Use $L(z)$ and the Fundamental Theorem of Calculus to compute the integral from part (a) by a different method.

Solution: Since the branch cut of $L(z)$ is on the negative imaginary axis, $L'(z) = 1/z$ at all points z on Γ. Therefore, the Fundamental Theorem of Calculus gives

$$\int_\Gamma \frac{1}{z} dz = L(-2) - L(2) = [\ln(2) + i\pi] - [\ln(2) + i \cdot 0] = i\pi.$$

2. Provide an example of an analytic function f with a pole of order 4 at $z_0 = 2i$ such that the residue $\mathrm{Res}(f; 2i) = -3$.

Solution: There are many answers to this question, but the simplest is probably

$$f(z) = \frac{1}{(z - 2i)^4} - \frac{3}{z - 2i},$$

which is already in Laurent series form.

3. Let f be analytic on the entire complex plane except at isolated singularities $z_1, \ldots, z_k$. Prove that f has an antiderivative if and only if $\mathrm{Res}(f; z_j) = 0$ for every $1 \leq j \leq k$.

Solution: Suppose $\text{Res}(f; z_j) = 0$ for all j. Let Γ be any simple closed contour such that every singularity z_j is either inside or outside Γ (that is, none of the z_j is actually on Γ). By the residue theorem,

$$\int_\Gamma f(z)dz = \pm 2\pi i \sum_{j:\, z_j \text{ is inside } \Gamma} \text{Res}(f; z_j) = 0,$$

where the $\pm$ is because Γ might be positively or negatively oriented. Since the integral of $f(z)$ around any closed loop is 0, a theorem in the textbook (Section 4.3, Theorem 7) implies that f has an antiderivative.

Conversely, suppose $\text{Res}(f; z_j) \neq 0$ for some particular j. If Γ is a small circle going counterclockwise around z_j,

$$\int_\Gamma f(z)dz = 2\pi i\, \text{Res}(f; z_j) \neq 0.$$

Therefore by the same textbook theorem, f cannot have an antiderivative.

4. Let C be the unit circle oriented counterclockwise, and define

$$f(z) = \int_C \frac{\sin(w)}{w - z} dw, \qquad g(z) = \int_C \frac{\sin(w)}{(w - z)^2} dw$$

for z not on the circle. Compute: $f(\pi/6), f(\pi/3), g(\pi/6), g(\pi/3)$.

Solution: If z is inside the unit circle, the Cauchy integral theorem says that

$$\frac{1}{2\pi i} \int_C \frac{\sin(w)}{w - z} dw = \sin(z),$$

$$\frac{d}{dz}\left[\frac{1}{2\pi i} \int_C \frac{\sin(w)}{w - z} dw\right] = \frac{1}{2\pi i} \int_C \frac{\sin(w)}{(w - z)^2} dw = \cos(z).$$

If z is outside the unit circle, the functions $\frac{\sin(w)}{w-z}$ and $\frac{\sin(w)}{(w-z)^2}$ (as functions of w) are analytic on and inside C, meaning that

$$\int_C \frac{\sin(w)}{w - z} dw = 0, \qquad \int_C \frac{\sin(w)}{(w - z)^2} dw = 0.$$

Thus: $f(\pi/6) = 2\pi i \sin(\pi/6) = \pi i$; $f(\pi/3) = 0$; $g(\pi/6) = 2\pi i \cos(\pi/6) = \sqrt{3}\pi i$; and $g(\pi/3) = 0$. (Note: The Residue theorem could also be used for this problem.)

5. Recall the Cauchy estimates: If f is analytic on and inside the circle C_R of radius R centered at z_0, and $|f(z)| \leq M$ for all z on C_R, then for all $n \geq 0$,

$$\left|f^{(n)}(z_0)\right| \leq \frac{n!M}{R^n}.$$

Suppose f is an entire function such that $|f(z)| \leq |z|^2$ for all $z \in \mathbf{C}$. Use the Cauchy estimates to prove that f must be a polynomial of degree at most 2, that is, $f(z) = c_0 + c_1 z + c_2 z^2$, and furthermore that $|c_2| \leq 1$. *Hint:* Consider the Taylor series for f centered at the origin.

Solution: Let $f(z) = \sum_{n=0}^{\infty} c_n z^n$ be the Taylor series for f centered at the origin. We have

$$c_n = \frac{f^{(n)}(0)}{n!}.$$

For any $R > 0$, since $|f(z)| \leq R^2$ when $|z| = R$, the Cauchy estimates give

$$|c_n| = \frac{|f^{(n)}(0)|}{n!} \leq \frac{R^2}{R^n}.$$

When $n \geq 3$ this upper bound tends to 0 as $R \to \infty$, so $c_n = 0$. When $n = 2$ we get $|c_2| \leq 1$. Thus the Taylor series simplifies to

$$f(z) = c_0 + c_1 z + c_2 z^2 + 0 + 0 + \cdots$$

with $|c_2| \leq 1$.

6. Compute the Laurent series for $f(z) = \dfrac{2}{z-1} + \dfrac{3}{z+5}$ in the annulus $\{2 < |z| < 3\}$. What is the largest annulus $\{r < |z| < R\}$ on which the Laurent series converges?

Solution: Since $2 < |z| < 3$,

$$\frac{2}{z-1} = \frac{2}{z} \cdot \frac{1}{1 - \frac{1}{z}} = \frac{2}{z} \sum_{n=0}^{\infty} \frac{1}{z^n}, \qquad \frac{3}{z+5} = \frac{3}{5} \cdot \frac{1}{1 + \frac{z}{5}} = \frac{3}{5} \sum_{n=0}^{\infty} (-1)^n \frac{z^n}{5^n}.$$

Therefore the Laurent series is

$$\sum_{n=1}^{\infty} \frac{2}{z^n} + \sum_{n=0}^{\infty} \frac{3(-1)^n}{5^{n+1}} z^n.$$

It converges on the largest possible annulus on which f has no singularities, which is $\{1 < |z| < 5\}$.

7. Find the singularities of $f(z) = \dfrac{e^{1/z}[\cos(z) - 1]}{(z+\pi)^2(z+2\pi)^4}$ and classify them as removable, essential, or poles. Find the order of each pole.

Solution: $f(z)$ has singularities at $z = 0, -\pi, -2\pi$. At $z = 0$,

$$f(z) = e^{1/z} \cdot \left[\frac{\cos(z) - 1}{(z+\pi)^2(z+2\pi)^4}\right],$$

where the first term has an essential singularity at 0 and the second term is analytic at 0. Therefore the singularity is essential.

At $z = -\pi$,

$$f(z) = \frac{1}{(z+\pi)^2} \cdot \left[\frac{e^{1/z}[\cos(z) - 1]}{(z+2\pi)^4}\right],$$

where the first term has a pole of order 2 at $-\pi$ and the second term is analytic and nonzero at $-\pi$. Therefore the singularity is a pole of order 2.

At $z = -2\pi$,

$$f(z) = \frac{\cos(z) - 1}{(z+2\pi)^4} \cdot \left[\frac{e^{1/z}}{(z+\pi)^2}\right],$$

where the second term is analytic and nonzero at -2π. To deal with the first term, we find the Taylor series for $g(z) = \cos(z)$ centered at -2π. The derivatives are $g^{(n)}(-2\pi) = g^{(n)}(0)$, by periodicity, so the Taylor series is

$$\cos(z) = 1 - \frac{(z+2\pi)^2}{2!} + \frac{(z+2\pi)^4}{4!} - \frac{(z+2\pi)^6}{6!} + \cdots.$$

Therefore,

$$\frac{\cos(z) - 1}{(z+2\pi)^4} = -\frac{1}{2!(z+2\pi)^2} + \frac{1}{4!} - \frac{(z+2\pi)^2}{6!} + \cdots$$

which means that f has a pole of order 2 at -2π.

8. What is the radius of convergence for the Taylor series of $f(z) = \dfrac{e^{\cos(z)}}{z^2+9}$ centered at -4?

Solution: The radius of convergence is the distance from -4 to the nearest singularity of f. Since f has singularities at $\pm 3i$, this gives a radius of $R = \sqrt{3^2 + 4^2} = 5$.

9. Suppose that $g_1(z)$ and $g_2(z)$ are both analytic at z_0. Also assume that $g_1(z_0) \neq 0$, while g_2 has a simple zero at z_0, so the Taylor series are

$$\begin{aligned} g_1(z) &= a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots \quad \text{with } a_0 \neq 0, \\ g_2(z) &= b_1(z - z_0) + b_2(z - z_0)^2 + \cdots \quad \text{with } b_1 \neq 0. \end{aligned}$$

Prove that $f(z) = g_1(z)/g_2(z)$ has a simple pole at z_0, and that

$$\operatorname{Res}(f; z_0) = \frac{a_0}{b_1} = \frac{g_1(z_0)}{g_2'(z_0)}.$$

Solution: First, we can read off from the Taylor series for g_1 and g_2 that $a_0 = g_1(z_0)$ and $b_1 = g_2'(z_0)$. To prove the first equality, write

$$\begin{aligned} f(z) &= \frac{a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots}{b_1(z - z_0) + b_2(z - z_0)^2 + \cdots} \\ &= \frac{1}{z - z_0} \left[\frac{a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots}{b_1 + b_2(z - z_0) + b_3(z - z_0)^2 + \cdots} \right]. \end{aligned}$$

The term in square brackets is analytic at z_0 and has the value a_0/b_1, so it has a Taylor expansion $c_0 + c_1(z - z_0) + c_2(z - z_0)^2 + \cdots$ where $c_0 = a_0/b_1$. Thus

$$f(z) = \frac{c_0}{z - z_0} + c_1 + c_2(z - z_0) + c_3(z - z_0)^2 + \cdots$$

has a simple pole at z_0 with residue $c_0 = a_0/b_1$.

10. Use residue theory to compute $\displaystyle\int_0^{2\pi} \frac{1}{13 + 12\cos\theta} d\theta$.
Hint: $(2z + 3)(3z + 2) = 6z^2 + 13z + 6$.

Solution: Let C be the unit circle positively oriented, parametrized by $z(\theta) = e^{i\theta}$, $0 \leq \theta \leq 2\pi$. We have $dz = z'(\theta)d\theta = ie^{i\theta}d\theta = izd\theta$, so $d\theta = dz/iz$. Thus

$$\begin{aligned} \int_0^{2\pi} \frac{1}{13 + 12\cos\theta} d\theta &= \int_C \frac{1}{13 + 12\left[\frac{1}{2}(z + \frac{1}{z})\right]} \cdot \frac{dz}{iz} = \frac{1}{i} \int_C \frac{1}{13z + 6z^2 + 6} dz \\ &= \frac{1}{i} \int_C \frac{1}{(2z + 3)(3z + 2)} dz. \end{aligned}$$

Denote the integrand by $f(z)$. We see that f has poles at $-2/3$ (inside the unit circle) and $-3/2$ (outside the unit circle), so

$$\frac{1}{i}\int_C \frac{1}{(2z+3)(3z+2)}dz = \frac{1}{i}\cdot 2\pi i\,\mathrm{Res}(f;-2/3).$$

We compute

$$\begin{aligned}\mathrm{Res}(f;-2/3) &= (z+2/3)f(z)\big|_{z=-2/3} = \left.\frac{z+2/3}{(2z+3)(3z+2)}\right|_{z=-2/3}\\ &= \left.\frac{1}{3(2z+3)}\right|_{z=-2/3} = \frac{1}{5}.\end{aligned}$$

Therefore the definite integral equals $2\pi/5$.

11. Compute the residue of $f(z) = \dfrac{1}{z^2+2z^3}$ at $z=0$.

Solution: Write

$$\begin{aligned}\frac{1}{z^2+2z^3} &= \frac{1}{z^2(1+2z)} = \frac{1}{z^2}\cdot\frac{1}{1+2z} = \frac{1}{z^2}\left[1-2z+4z^2-8z^3+\cdots\right]\\ &= \frac{1}{z^2}-\frac{2}{z}+4-8z+\cdots,\end{aligned}$$

so the residue of f at 0 is -2.

Alternate solution: Since f has a pole of order 2 at 0, use the formula

$$\mathrm{Res}(f;0) = \frac{1}{(2-1)!}\frac{d^{2-1}}{dz^{2-1}}\left[z^2f(z)\right]\Big|_{z=0} = \left.\frac{-2}{(1+2z)^2}\right|_{z=0} = -2.$$

12. Use residue theory to compute p.v. $\displaystyle\int_{-\infty}^{\infty}\frac{\sin(2x)}{x+i}dx$.

Solution: Write $\sin(2x) = \frac{1}{2i}\left[e^{2ix}-e^{-2ix}\right]$, so we want to compute

$$\frac{1}{2i}\left[\text{p.v.}\int_{-\infty}^{\infty} e^{2ix}\cdot\frac{1}{x+i}dx \quad - \quad \text{p.v.}\int_{-\infty}^{\infty} e^{-2ix}\cdot\frac{1}{x+i}dx\right].$$

Let (I) denote the first integral and (II) denote the second integral, so we want $\frac{1}{2i}[(\mathrm{I})-(\mathrm{II})]$.

For $\rho > 0$, let γ_ρ be the line segment going from $-\rho$ to ρ along the real axis; let C_ρ^+ be the upper semicircle going from ρ to $-\rho$ counterclockwise along the circle $|z| = \rho$; and let C_ρ^- be the lower semicircle going from ρ to $-\rho$ clockwise along the same circle. Define the closed contours $\Gamma_\rho^+ = \gamma_\rho + C_\rho^+$ and $\Gamma_\rho^- = \gamma_\rho + C_\rho^-$.

To compute (I), we know that

$$\int_{\Gamma_\rho^+} e^{2iz} \cdot \frac{1}{z+i} dz = 0$$

since there are no singularities inside the contour. Also,

$$\lim_{\rho\to\infty} \int_{\gamma_\rho} e^{2iz} \cdot \frac{1}{z+i} dz = \text{(I)},$$

$$\lim_{\rho\to\infty} \int_{C_\rho^+} e^{2iz} \cdot \frac{1}{z+i} dz = 0 \quad \text{by Jordan's Lemma.}$$

Details of the use of Jordan's Lemma: For large enough ρ, if z is on C_ρ^+,

$$\left|\frac{1}{z+i}\right| = \left|\frac{1}{z}\right| \cdot \left|\frac{1}{1+i/z}\right| \leq \frac{1}{\rho} \cdot 2.$$

Therefore,

$$\left|\int_{C_\rho^+} e^{2iz} \cdot \frac{1}{z+i} dz\right| \leq \frac{\pi \cdot \frac{2}{\rho}}{2},$$

which has a limit of 0 as $\rho \to \infty$.

Putting the pieces together, we have $\text{(I)} + 0 = 0$.

To compute (II), we know that

$$\int_{\Gamma_\rho^-} e^{-2iz} \cdot \frac{1}{z+i} dz = -2\pi i \,\text{Res}(-i),$$

since Γ_ρ^- is negatively oriented, and $\text{Res}(-i) = e^{-2iz}|_{z=-i} = e^{-2}$. Also,

$$\lim_{\rho\to\infty} \int_{\gamma_\rho} e^{-2iz} \cdot \frac{1}{z+i} dz = \text{(II)},$$

$$\lim_{\rho\to\infty} \int_{C_\rho^-} e^{-2iz} \cdot \frac{1}{z+i} dz = 0 \quad \text{by Jordan's Lemma.}$$

Therefore, (II) + 0 $= -2\pi i e^{-2}$.

Finally, the integral we are trying to compute is

$$\text{p.v.} \int_{-\infty}^{\infty} \frac{\sin(2x)}{x+i} dx = \frac{1}{2i} [(\text{I}) - (\text{II})] = \frac{1}{2i} \left[2\pi i e^{-2}\right] = \frac{\pi}{e^2}.$$